A geometry of 7-point planes

(Material Based on [Tay09, Chapter 6, pp 43-45])

(Wednesday 16th March 2014)

Abstract

The classification of finite simple groups gives four families of groups whose union is precisely the set of all finite simple groups. However, when are two such groups isomorphic to one another?

In this talk we will show that the alternating group of degree 8 is isomorphic to the matrix group PSL(4,2) by considering group actions on a geometry of 7-point planes. No prior knowledge of geometries is required.

1 Motivation

We begin with everyone's favourite mathematical theorem:

Theorem 1 (The Classification Theorem of Finite Simple Groups)**.** *Every finite simple group is isomorphic to one of the following:*

(i) A cyclic group of prime order;

(ii) A_n *for some* $n \geq 5$*;*

(iii) A simple group of lie type - these include lots of matrix groups such as projective special linear, unitary, symplectic, or orthogonal groups over a finite field; or

(iv) one of 26 sporadic groups.

The classification theorem has had a profound effect on finite group theory. Indeed, many important theorems have now been proved using it. However, there is repetition within the theorem. Indeed we have the isomorphisms (see [Wil09] for more details).

$$\begin{array}{ccccc} PSL_2(4) & \cong & PSL_2(5) & \cong & \mathrm{Alt}(5) \\ & & PSL_2(7) & \cong & PSL_3(2) \\ & & PSL_2(9) & \cong & \mathrm{Alt}(6) \\ & & PSL_4(2) & \cong & \mathrm{Alt}(8) \\ & & PSU_4(2) & \cong & PSp_4(3) \end{array}$$

In this talk we will prove the penaultimate isomorphism by considering finite geometries.

2 Finite Geometries

We begin with the definition of a geometry.

Definition 2. *A* geometry over a set Δ *is a triple* $(\Gamma, *, \tau)$*, where;*

- Γ *is a set (the* elements *of* Γ*);*
- $*$ *is a symmetric relation on* Γ *(the* incident relation *of* Γ*);*
- $\tau : \Gamma \twoheadrightarrow \Delta$ *is a surjective map (the* type function *of* Γ*).*

such that if $x, y \in \Gamma$ *and* $x * y$*, then* $\tau(x) \neq \tau(y)$*.*

We often talk about the geometry Γ, rather than $(\Gamma, *, \tau)$. Geometries arise in many different areas of mathematics.

Example 3. *Consider a cube, having vertex set* V*, edge set* E *and face set* F*. Defining* $\Gamma = V \cup E \cup F$*,* $*$ *by*

$$a * b \Leftrightarrow a \in b \text{ or } b \in a$$

and $\tau : \Gamma \twoheadrightarrow \Delta := v, e, f$ *by*

$$\tau x := \begin{cases} v & \text{if } x \in V; \\ e & \text{if } x \in E; \text{ and} \\ f & \text{if } x \in F, \end{cases}$$

then $\Gamma = (\Gamma, *, \tau)$ *is a geometry over* Δ*.* •

Example 4. *Let* G *be a finite group,* p *a prime with* $p||G|$*. Let* Γ *be the set of all chains of* p*-subgroups of* G*, and suppose the maximal length of such a chain is* m *(so this is just the order complex of the Brown complex). Define* $\Delta := \{1, \ldots, m\}$ *and define the natural map* $\tau : \Gamma \to \Delta$ *by* $\tau(x) = \text{length}(x)$*. By adding the relation* $*$ *defined to be inclusion, we have that* $\Gamma = (\Gamma, *, \tau)$ *is a geometry over* Δ*.* •

Example 4 is a specific case of a simplicial complex, and clearly any simplicial complex of dimension d can be considered as a geometry over $\{1, \ldots, d + 1\}$ in the natural way.

We now consider a specific geometry.

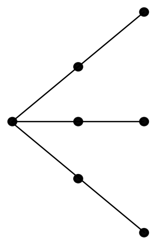
3 7-point planes

Let Ω be a set of seven elements, and denote by $\mathcal{L}$ the set of 3-element subsets of Ω. Our aim is to identify the elements of $\mathcal{L}$ with lines of a projective geometry. Hence we call elements of $\mathcal{L}$ Lines (with a capital l!)

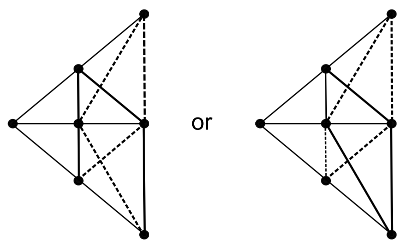
Definition 5. *A* 7-point plane *is a set $\mathbb{B}$ of seven Lines such that any two Lines of $\mathbb{B}$ have exactly one element of Ω in common.*

Definition 6. *A* fan with vertex $\alpha \in \Omega$ *is a set of three Lines such that any pair of Lines have only α in common.*

We may think of fans diagrammatically as



For each $\alpha \in \Omega$, there are $\binom{6}{2,\,2,\,2}/6 = 15$ fans on α, and hence there are 105 fans. Moreover, each fan can be extended to a seven point plane in exactly two ways; either



Every 7-point plane contains a fan with vertex α, for each $\alpha \in \Omega$ *(as there are $\binom{7}{2} = 21$ common elements of the seven lines in the plane, so 3 pairs of distinct lines each have common element α for a given $\alpha \in \Omega$).* It follows that the set $\mathcal{A}$ of 7-point planes on Ω has cardinality 30.

Clearly $Sym(7)$ acts on Ω, and hence permutes the 7-point planes. Since the subgroup of $Sym(7)$ which fixes a 7-point plane is $PSL_3(2)$ (we just assume this) and $[Sym(7) : PSL_3(2)] = 30$, we see[1] that the action of $Sym(7)$ on $\mathcal{A}$ is transitive. Similarly as $PSL_3(2) \subset Alt(7)$ and

[1] using the Orbit-Stabilizer Theorem, as $[Sym(7) : \mathrm{Stab}(x)] = |\mathrm{Orbit}(x)|$ for all $x \in \Omega$.

$[Alt(7) : PSL_3(2)] = 15$, we see that $Alt(7)$ has two orbits on $\mathcal{A}$, each of length 15.

Let $\mathcal{P}$ and $\mathcal{H}$ denote the two orbits of $Alt(7)$ on $\mathcal{A}$. we call the elements of $\mathcal{P}$ and $\mathcal{H}$, $\underline{\text{P}}$oints and $\underline{\text{P}}$lanes respectively. we wish to define a geometry from $\mathcal{P}, \mathcal{L}$ and $\mathcal{H}$ and then prove that these sets are the points, lines and planes of a projective geometry of a vector space of dimension 4 over $GF(2)$.

Set $\Delta := \{P, L, H\}$, $\Gamma = \mathcal{P} \sqcup \mathcal{L} \sqcup \mathcal{H}$ and $\tau : \Gamma \twoheadrightarrow \Delta$ in the obvious way:

$$\tau(\sigma) := \begin{cases} P & \text{if } \sigma \in \mathcal{P} \\ L & \text{if } \sigma \in \mathcal{L} \\ H & \text{if } \sigma \in \mathcal{H}. \end{cases}$$

We then define an incidence relation $*$ as follows; a $\underline{\text{L}}$ine $\lambda \in \mathcal{L}$ is incident to a $\underline{\text{P}}$oint or $\underline{\text{P}}$lane $\mathbb{B} \in \mathcal{P} \cup \mathcal{H}$ whenever $\lambda \in \mathbb{B}$; the $\underline{\text{P}}$oint $\mathbb{B}_1$ is incident with the $\underline{\text{P}}$lane $\mathbb{B}_2$ whenever $\mathbb{B}_1 \cap \mathbb{B}_2$ is a fan. Thus $(\Gamma, *, \tau)$ is a geometry over Δ. To identify the geometry Γ, we proceed to investigate the incident relation $*$ in some detail. First we note that $Alt(7)$ acts transitively on each of the sets $\mathcal{P}, \mathcal{L}$ and $\mathcal{H}$, and preserves $*$ (*since it merely relabels the elements of* Ω). Moreover, the elements of $Sym(7) \setminus Alt(7)$ interchange $\mathcal{P}$ and $\mathcal{H}$. We are now in a position to prove our first result:

Lemma 7. *(i) Each $\underline{\text{L}}$ine is incident to 3 $\underline{\text{P}}$oints and 3 $\underline{\text{P}}$lanes.*
(ii) A $\underline{\text{P}}$oint $\mathbb{B}_1$ is incident to a $\underline{\text{P}}$lane $\mathbb{B}_2$ if and only if $\mathbb{B}_1$ and $\mathbb{B}_2$ are incident to a common line.

Proof. (i) Let λ be a $\underline{\text{L}}$ine, and let $\mathbb{B} \in \mathcal{A}$ contain λ. Each $\alpha \in \lambda$ belongs to two $\underline{\text{L}}$ines of $\mathbb{B}$ other than λ and these $\underline{\text{L}}$ines create a partition of $\Omega \setminus \lambda$ into two sets of size 2. Hence we have a bijection between the 3 elements of λ and the 3 partitions of $\Omega \setminus \lambda$ into two sets of size 2. Conversely, every such bijection arises from a unique 7-point plane which contains λ *(to see this just try constructing such a plane)*. Since there are 6 such bijections *(since a bijection is from an set of cardinality 3 to another set of cardinality 3, so is an element of* $Sym(3)$*)*, we deduce that λ is contained in six 7-point planes.

Let L denote the stabilizer of λ in $Sym(7)$. Then each element of L permutes the 3 elements of λ, and also permutes the 4 elements of $\Omega \setminus \lambda$. Thus $L = R \times T$, where $R \cong Sym(3)$ and $T \cong Sym(4)$. Since L fixes λ, it acts on the six 7-point planes containing λ. Moreover, the transpositions of L interchange the 7-point planes of $\mathcal{P}$ with those of $\mathcal{H}$. Thus λ is incident to 3 $\underline{\text{P}}$oints and 3 $\underline{\text{P}}$lanes.

(ii) Clearly the given condition is necessary for $\mathbb{B}_1$ and $\mathbb{B}_2$ to be incident. To see that it is also sufficient, assume $\mathbb{B}_1$ and $\mathbb{B}_2$ are incident to a common line λ. By (i), there are three $\underline{\text{P}}$oints and three $\underline{\text{P}}$lanes incident to λ. Moreover, every transposition of L fixes a fan containing λ (*possibly* *illustrate this on the board*). Each of the 3 transpositions of R maps $\mathbb{B}_1$ to a $\underline{\text{P}}$lane containing λ, and there are three such $\underline{\text{P}}$lanes, one of which is $\mathbb{B}_2$. Hence B_1 is incident to $\mathbb{B}_2$. $\square$

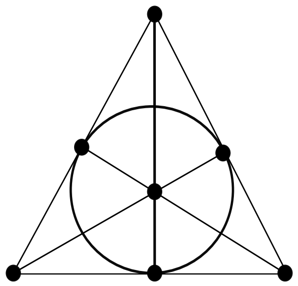
We now relate this geometry to a certain projective geometry.

Definition 8. *If V is a finite-dimensional vector space, then the set of all subspaces of V, partially ordered by inclusion, is called the* projective geometry of V, *denoted $\mathcal{P}(V)$.*

Theorem 9. *The $\underline{\text{P}}$oints $\mathcal{P}$, $\underline{\text{L}}$ines $\mathcal{L}$ and $\underline{\text{P}}$lanes $\mathcal{H}$ can be identified with the points (1-dimensional subspace), lines (2-dimensional subspaces) and planes (3-dimensional subspaces) of the projective geometry of a vector space of dimension 4 over $GF(2)$, so that the incidence relation $*$ becomes the usual containment of subspaces.*

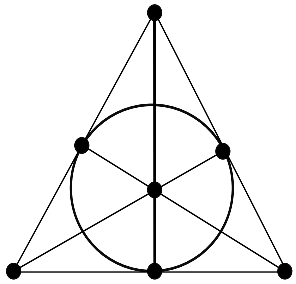
Proof. If $\mathbb{B}$ is a $\underline{\text{P}}$oint, there are 7 $\underline{\text{L}}$ines incident to $\mathbb{B}$. Each of these $\underline{\text{L}}$ines will be incident to a further two $\underline{\text{P}}$oints. Moreover, any two $\underline{\text{P}}$oints can be incident to at most one common $\underline{\text{L}}$ine *(since any two lines of a $\underline{\text{P}}$oint by definition have a common element α, so we can extend these two $\underline{\text{L}}$ines to a flag at α, and any flag can be extended to a unique $\underline{\text{P}}$oint).* Thus we have accounted for all 15 $\underline{\text{P}}$oints. Furthermore, each pair of distinct $\underline{\text{P}}$oints is incident to a unique $\underline{\text{L}}$ine. An analogous argument shows that every pair of $\underline{\text{P}}$lanes are incident to a unique $\underline{\text{L}}$ine. Also, since there are 7 flags in $\mathbb{B}$ and each flag is incident to a unique $\underline{\text{P}}$lane, there are 7 $\underline{\text{P}}$lanes incident to $\mathbb{B}$.

Let $V = \mathcal{P} \cup \{0\}$ and let $\mathbb{B}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{P}$ ($\mathbb{B}_1 \neq \mathbb{B}_2$). Define $\mathbb{B} + \mathbb{B} = 0$ and $0 + \mathbb{B} = \mathbb{B} + 0 = \mathbb{B}$, and define $\mathbb{B}_1 + \mathbb{B}_2$ to be the third $\underline{\text{P}}$oint incident to the unique $\underline{\text{L}}$ine determined by $\mathbb{B}_1$ and $\mathbb{B}_2$. Any two $\underline{\text{L}}$ines with a common $\underline{\text{P}}$oint determine a fan at the common element, and hence are incident to a common $\underline{\text{P}}$lane, H. This in turn implies associativity of addition, and hence V is a $GF(2)$-vector space.
To see this, we may note that H is incident to 7 $\underline{\text{L}}$ines and 7 $\underline{\text{P}}$oints, such that each $\underline{\text{L}}$ine is incident to 3 $\underline{\text{P}}$oints and vice versa. Thus representing $\underline{\text{P}}$oints as dots in the following diagram of the $\underline{\text{P}}$lane H, we see that



It is then clear that associativity holds. By construction, the sets $\mathcal{P}, \mathcal{L}$ and $\mathcal{H}$ of all $\underline{\text{P}}$oints, $\underline{\text{L}}$ines and $\underline{\text{P}}$lanes correspond to the 1-,2- and 3-dimensional subspaces of V, namely the points, lines and planes of the projective geometry of V. $\square$

It is sometimes useful to represent a 7-point plane, $\mathbb{B}$, diagrammatically as:



By labeling the elements of Ω as $1, \ldots, 7$, we can see that the stabiliser of an incident $\underline{\text{P}}$oint-$\underline{\text{L}}$ine (or $\underline{\text{P}}$lane-$\underline{\text{L}}$ine) pair has order 24 (since you may permute the elements of the line in any way, and then the $\underline{\text{P}}$oint is uniquely determined by the positioning of one further element of Ω). Since $|Alt(7)|/24 = 105$ and there are $15 \times 7 = 105$ incident $\underline{\text{P}}$oint-$\underline{\text{Line}}$ ($\underline{\text{P}}$lane-$\underline{\text{L}}$ine) pairs, it follows that $Alt(7)$ acts transitively on such pairs.

Theorem 10. *We have $Alt(7) \subset PGL_4(2)$ and $Alt(7)$ acts 2-transitively on both the points and planes of the projective geometry of dimension 4 (*the book says dimension 3*) over $GF(2)$.*

Proof. We know that $Alt(7)$ acts transitively on $\mathcal{P}$, and two distinct elements of $Alt(7)$ have distinct actions on $\mathcal{P}$. Since the set $\mathcal{P}$ of $\underline{\text{P}}$oints correspond to the points of the projective geometry of dimension 4 over $GF(2)$, it follows that $Alt(7) \subset PGL_4(2)$. Let $\mathbb{B}_i$ (for $i = 1, 2, 3, 4$) be $\underline{\text{P}}$oints, and let λ and λ' be the lines determined by the pairs $(\mathbb{B}_1, \mathbb{B}_2)$ and $(\mathbb{B}_3, \mathbb{B}_4)$. Since $Alt(7)$ acts transitively on $\underline{\text{P}}$oint-$\underline{\text{L}}$ine pairs, there exists $\sigma \in Alt(7)$ such that σ maps $(\mathbb{B}_1, \lambda)$ to $(\mathbb{B}_3, \lambda')$. Moreover, σ maps $\mathbb{B}_2$ to some $\underline{\text{P}}$oint $\mathbb{B}'_4$ incident to λ'. The stabilizer in $Alt(7)$ of λ' acts as $Sym(3)$ on the $\underline{\text{L}}$ine λ'. Hence as there are three $\underline{\text{P}}$oints incident to λ', there exists $\gamma \in Alt(7)$ that maps $(\mathbb{B}_3, \mathbb{B}'_4)$ to $(\mathbb{B}_3, \mathbb{B}_4)$. It follows that $Alt(7)$ acts 2-transitively on $\mathcal{P}$ and hence on the points of the projective geometry. By symmetry, $Alt(7)$ is 2-transitive on $\mathcal{H}$ as well. $\square$

Corollary 11. $GL_4(2) = SL_4(2) = PSL_4(2) = PGL_4(2) \cong Alt(8)$.

Proof. We know that $[PSL_4(2) : Alt(7)] = 8$ ($|PSL_4(2)| = 20160, |Alt(7)| = 2520$ *and* $|Sym(8)| = 40320$), and $PSL_4(2)$ acts on the 8 cosets of $Alt(7)$ in $PSL_4(2)$. It follows that we may define a homomorphism $f : PSL_4(2) \longrightarrow Sym(8)$. As $PSL_4(2)$ is simple, ker f is trivial, so we may consider $PSL_4(2) \subset Sym(8)$. As $[PSL_4(2) : Sym(8)] = 2$, and $Alt(8)$ is the only subgroup of $Sym(8)$ of index 2, we must have $SL_4(2) \cong Alt(8)$. $\square$

References

[Tay09] Donald E. Taylor. *The Geometry of the Classical Groups*, volume 9 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Lemgo, Germany, 2009.

[Wil09] Robert A. Wilson. *The Finite Simple Groups*, volume 251 of *Graduate Texts in Mathematics*. Springer-Verlag, London, United Kingdom, 2009.