Sparse Tensor FEM for Operator Equations with Stochastic Data

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Abstract:

Let $A\colon V \to V'$ be a linear, strongly elliptic operator on a d-dimensional manifold D (polyhedra or boundaries of polyhedra are also allowed). An operator equation $Au = f$ with stochastic data f is considered. The goal of the computation is the mean field and k-point correlation functions $\mathcal{M}^1 u \in V$, $\mathcal{M}^2 u \in V \otimes V$, ...,$\mathcal{M}^k u \in V \otimes \cdots \otimes V$ of the random solution u.

We discretize the mean field problem using an h-Version FEM in D with hierarchical basis and N degrees of freedom. We analyze a Monte-Carlo (MC) algorithm and a deterministic algorithm for the approximation of the moment $\mathcal{M}^k u$ for $k \geq 1$.

Both algorithms are based on a "sparse tensor product" of a multilevel Finite Element space for the approximation of $\mathcal{M}^k u$ with $O\left(N(\log N)^{k-1}\right)$ degrees of freedom, instead of N^k degrees of freedom for the full tensor product of the FE space.

Algorithm 1, a sparse tensor MC-FEM with M samples (i.e., deterministic solves) is proved to yield approximations to $\mathcal{M}^k u$ with work of $O(MN(\log N)^{k-1})$ operations. The solutions are shown to converge with the optimal rates with respect to the number of Finite Element degrees of freedom N and the number M of samples.

Algorithm 2, the deterministic FEM, is based on deterministic, hypoelliptic equations for the kth moment $\mathcal{M}^k u$ of the random solution u in $D^k \subset \mathbb{R}^{kd}$. We analyze their discretization using sparse tensor products of the hierarchic FE spaces in D.

For nonlocal operators A wavelet compression is employed in both algorithms. The linear systems are solved iteratively with multilevel preconditioning. We prove that this yields an approximation for $\mathcal{M}^k u$ with at most $O\left(N(\log N)^{k+1}\right)$ operations.

For nonlinear problems, we employ a linearization to derive deterministic equations for the statistics of the random solution in terms of the statistics of the input data. An example for elliptic problems in domains with small amplitude stochastic boundary perturbation based on a second order shape calculus and a boundary integral reformulation of the hypoelliptic system of the system for the second moments of the random solution.