A local restart technique for symmetric nonlinear eigenvalue problems

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Joint work with Heinrich Voss and Vera Lochmann

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Noise radiation

The simulation of the sound radiation requires the solution of the Helmholtz equation

$$\Delta u + k^2 u = 0$$

on the exterior of the domain occupied by the tire for a particular set of wave numbers $k = \omega_k / c$, c - speed of sound.

ω_k are obtained by Fourier analysis of the transient excitation forces $f(t) \approx \sum_k f_k e^{i\omega_k t}$ such as an impact of the roughness of the road surface or the tread pattern.

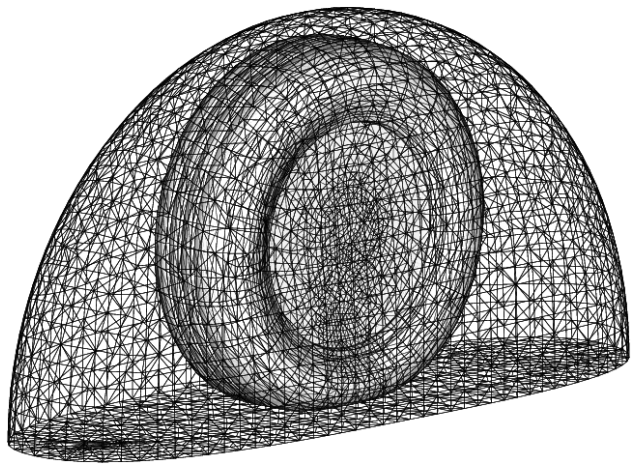

Figure: Acoustic model of a tire.

Boundary conditions

Sommerfeld radiation condition

$$\left| \frac{\partial u}{\partial r} - iku \right| \leq \frac{C}{r^2} \tag{1}$$

is assumed for large r.

On the outer boundary of the tire the normal velocities of the tire for the particular frequency ω_k are prescribed as an excitation.

Discretization of ALE model by finite elements yields the transient equation of motion

$$M\ddot{u} + G\dot{u} + Ku = f(t), \tag{2}$$

where $M = M^T$ is the mass matrix, $G = -G^T$ is the gyroscopic matrix and $K = K^T$ is the stiffness matrix modified by the inertia forces due to the stationary rolling.

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Gyroscopic eigenvalue problem

Assuming the modal superposition $u_k \approx \tilde{u}_k = Xz_k$, an approximation to the normal velocities for the particular frequency ω_k can be extracted from the projected equation

$$-\omega_k^2 X^* M X z_k + i\omega_k X^* G X z_k + X^* K X z_k = X^* f_k,$$

where the right hand side is a vector of spectral amplitudes of modal excitation forces for a frequency ω_k.

Gyroscopic eigenvalue problem

$$-\lambda^2 Mx + i\lambda Gx + Kx = 0,$$

In practice only the eigenmodes with frequencies close to the sought frequency ω_k are used which corresponds to the range up to 2000 Hz.

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Outline

Nonlinear Eigenvalue problem

We consider a nonlinear eigenvalue problem

$$T(\lambda)x = 0$$

where $T(\lambda) \in \mathbb{C}^{nxn}$ large, sparse and $T(\lambda) = T(\lambda)^*$ for every λ in an open real interval J.

We assume that for every fixed $x \neq 0$ the real function

$$f(\lambda; x) := x^* T(\lambda) x$$

is continuously differentiable on J, and that the equation

$$f(\lambda; x) = 0 \tag{3}$$

has at most one solution in J.

Definition

Then equation (3) implicitly defines a functional p on some subset D of $\mathbb{C}^n \setminus \{0\}$ which we call the Rayleigh functional.

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Eigenvalue enumeration

In general the natural enumeration is not appropriate but:

Lemma

If λ is an eigenvalue of $T(\lambda)x = 0$ then $\mu = 0$ is an eigenvalue of $T(\lambda)y = \mu y$.

Therefore $\exists k \in \mathbb{N}$ such that

$$0 = \max_{\mathcal{W} \in S_k} \min_{w \in \mathcal{W}^1} w^* T(\lambda) w$$

S_k: set of all k-dimensional subspaces of $\mathbb{C}^n$

$\mathcal{W}^1 := \{w \in \mathcal{W} : \|w\| = 1\}$ is the unit sphere in $\mathcal{W}$.

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In this case we call λ a kth eigenvalue of $T(\lambda)x = 0$.

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Minmax characterization

Theorem (Voss, Werner 82)

Let $p : D \to J$ be the Rayleigh functional and assume that $x^* T'(p(x))x > 0$ for every $x \in D$. Then it holds:

(i) *For every $k \in \mathbb{N}$ there is at most one k-th eigenvalue of problem $T(\lambda)x = 0$ which can be characterized by*

$$\lambda_k = \min_{\substack{\mathcal{W} \in S_k, \\ \mathcal{W} \cap D \neq \emptyset}} \sup_{w \in \mathcal{W} \cap D} p(w). \tag{4}$$

Hence, there are at most n eigenvalues of $T(\lambda)x = 0$ in J.

(ii) *If $\lambda \in J$ and $k \in \mathbb{N}$ such that kth eigenvalue of $T(\lambda)x = 0$, $\lambda_k \in J$. Then it holds*

$$\lambda \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \lambda_k \quad \Longleftrightarrow \quad \mu_k(\lambda) := \max_{\mathcal{W} \in S_k} \min_{w \in \mathcal{W}^1} w^* T(\lambda) w \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0.$$

Safeguarded Iteration

Safeguarded iteration is a fix-point iteration. It aims for an eigenvalue with a particular number k

Require: α_1 an approximation to the kth eigenvalue
  for $i = 1, \ldots$ until convergence **do**
    Determine an eigenvector x_i corresponding to the kth largest eigenvalue of $T(\alpha_i)$
    Solve $x_i^* T(\alpha_{i+1}) x_i = 0$ for α_{i+1}
  end for

Convergence properties:

- Global quadratic convergence to $\lambda_1 \in J$
- If λ_k is an simple eigenvalue local quadratic convergence to λ_k
- If $T'(\lambda)$ is positive definite and x_i is replaced by the kth eigenvector of $T(\alpha_i)x = \kappa T'(\alpha_i)x$ then the convergence is even cubic.

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Subspace expansion

- Jacobi Davidson
- Residual Inverse Iteration

$$t = T(\sigma)^{-1} T(\lambda_k) u_k \tag{5}$$

For the linear problem $T(\lambda) = A - \lambda B$ this is exactly a Cayley transform

$$(A - \sigma B)^{-1}(A - \lambda B) = I + (\sigma - \lambda)(A - \sigma B)^{-1} B$$

and therefore equivalent to the shift-invert Arnoldi method.

If the linear system $T(\sigma)t = T(\lambda)u$ is too expensive to solve we can choose a new direction as

$$t = M^{-1} T(\lambda) u \quad \text{with} \quad M \approx T(\sigma).$$

For the linear problem this again corresponds to the inexact Cayley transform.

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Require: $m \geq 1, \quad V, \quad V^*V = I, \quad \sigma, \quad M \approx T(\sigma)^{-1}$
  while $m \leq$ number of wanted eigenvalues **do**
    Compute mth eigenpair (μ, y) of $T_V(\cdot)y := V^*T(\cdot)Vy$
    $u = Vy$ and $r = T(\mu)u$
    if $\|r\|/\|u\| < \epsilon$ **then**
      ACCEPT $(\lambda_m, x_m) = (\mu, u)$
      $m = m + 1$
      Update σ and $M \approx T(\sigma)^{-1}$ **if** necessary
      RESTART if necessary
    end if
    $v = Mr, \quad v = v - VV^*v, \quad \tilde{v} = v/\|v\|, \quad V = [V, \tilde{v}]$
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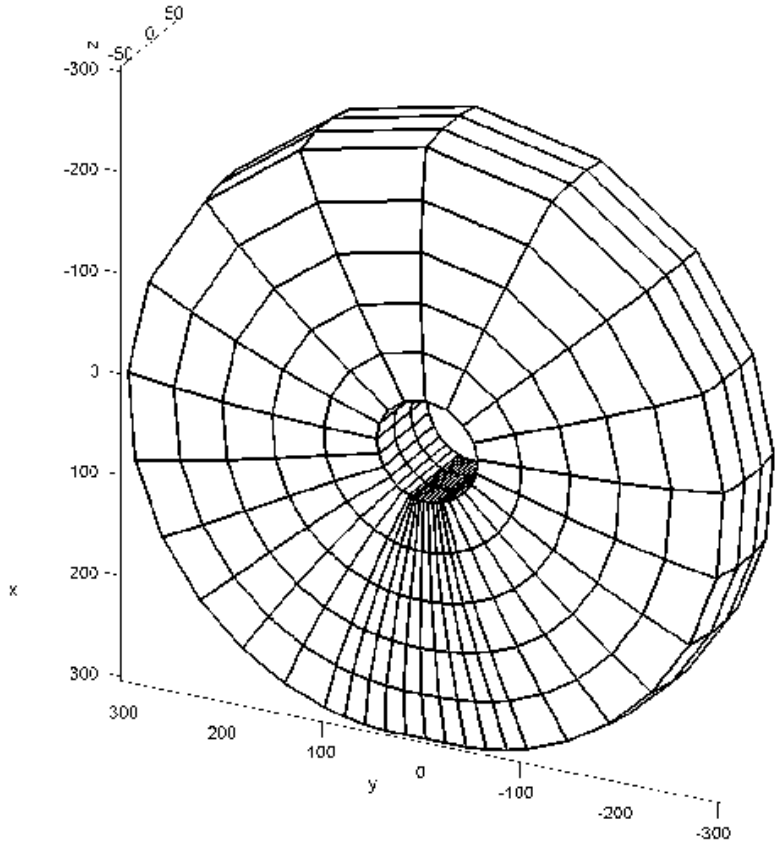
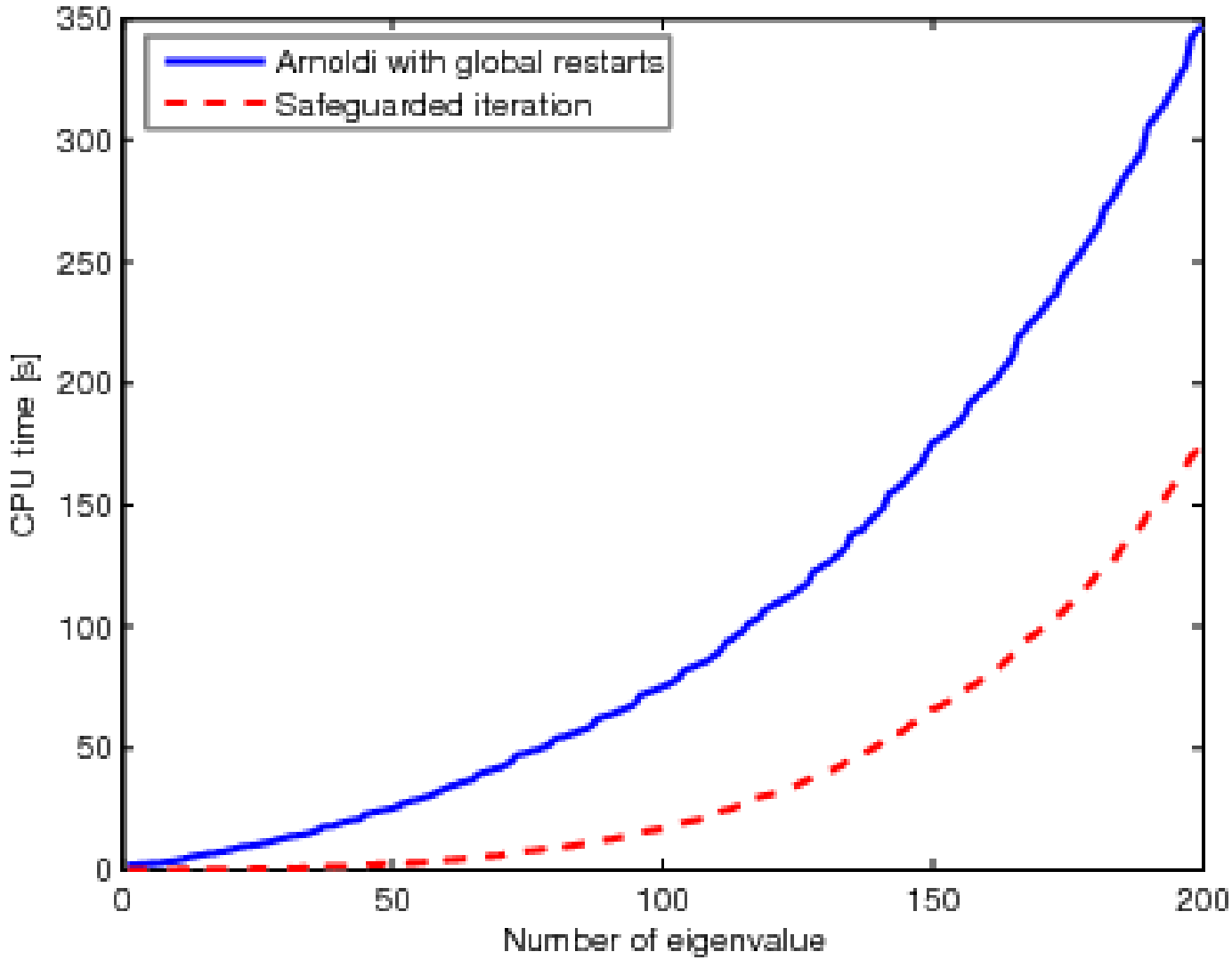
Solid rubber wheel

The wheel is pressed on the track and is rotating at a rate of 50Hz (1728 DOFs).



Nonlinear Arnoldi with global restarts



Local numbering

Let $(\hat{\lambda}, \hat{x})$ an eigenpair of $T(\lambda)x = 0$. We call $\hat{\lambda}$ an anchor.

Let $\mathcal{V} \subset \mathbb{C}^n : \hat{x} \in \mathcal{V}$ and V be its orthogonal basis.

Then $\hat{\lambda}$ is also an eigenvalue of the projected problem

$$T_V(\hat{\lambda})y := V^* T(\hat{\lambda}) V y = 0. \tag{6}$$

Since $T_V(\cdot)$ satisfies the conditions of Theorem 4 we can assign to $\hat{\lambda}$ a local number $\ell = \ell(\mathcal{V})$ in the following way:

$\hat{\lambda}$ is an ℓth eigenvalue of problem $T_V(\lambda)y = 0$ if $\mu(\hat{\lambda}) = 0$ is the ℓ largest eigenvalue of the linear problem

$$T_V(\hat{\lambda})y = \mu(\hat{\lambda})y. \tag{7}$$

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In the course of iterative method subspaces are generated

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset V_k.$$

In kth step enumerate the eigenvalues within $\mathcal{V}_k$ locally from the anchor

$$\hat{\lambda} =: \lambda_{\ell(\mathcal{V}_k)}, \lambda_{\ell(\mathcal{V}_k)+1}, \ldots$$

Generate $\mathcal{V}_{k+1}$ aiming at the $(\ell(\mathcal{V}_k) + 1)$st eigenvalue

After convergence we may continue the iterative projection method aiming at the $(\ell(\mathcal{V}_{k'}) + 2)$nd eigenvalue.

Or we may replace the anchor by the newly converged eigenpair.

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Restart framework

Require: Preconditioner $M \approx T(\sigma)^{-1}$ for a suitable shift σ,
Require: (λ_i, x_i) an (approximate) eigenpair of $T(\cdot)$
Require: v_1 an approximation to x_{i+1}

$V = [x_i, v_1]$;
j = 1;
while Restart condition not satisfied **do**
  repeat
    Determine largest eigenvalues $\mu_1(\lambda_i) \geq \cdots \geq \mu_k(\lambda_i) > 0 \geq \mu_{k+1}(\lambda_i)$ of $T_V(\lambda_i)$
    Set $\ell := k$ if $\mu_k \leq -\mu_{k+1}$, and else $\ell := k + 1$
    Compute $(\ell + j)$th eigenpair $(\tilde{\lambda}_{\ell+j}, y_{\ell+j})$ of $T_V(\cdot)$
    Expand V aiming at $(\lambda_{\ell+j}, x_{\ell+j})$
  until Eigenpair $(\tilde{\lambda}_{\ell+j}, Vy_{l+j}) =: (\lambda_{i+j}, x_{i+j})$ converged
  j = j+1;
end while

Spurious eigenvalues

Eigenvalues of $T(\cdot)$ in $[\lambda_i, \lambda_{i+j}]$:

$$\hat{\lambda} =: \lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+j}, \lambda_{n+j+1}$$

Eigenvalues of $T_V(\cdot)$ in $[\lambda_i, \lambda_{i+j}]$ ideally:

$$\hat{\lambda} =: \lambda_{\ell(\nu_k)}, \lambda_{\ell(\nu_k)+1}, \ldots, \lambda_{\ell(\nu_k)+j}, \lambda_{\ell(\nu_k)+j+1}$$

But it may happen

$$\hat{\lambda} =: \lambda_{\ell(\nu_k)}, \lambda_{\ell(\nu_k)+1}, \ldots, \theta =: \lambda_{\ell(\nu_k)+m}, \ldots, \lambda_{\ell(\nu_k)+j}, \lambda_{\ell(\nu_k)+j+1}$$

Spurious eigenvalues temporarily arise in the search subspace which are linear combinations of the eigenvalues outside of the part of the spectrum covered by $\mathcal{V}_k$.

Local number of λ_{i+j} is raised by 1, and λ_{i+j} is accepted as an $(i+j+1)$st eigenvalue.

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Workaround

In the course of the computation it may happen that the algorithm converges to an eigenvalue twice

$$\lambda_i < \lambda_{i+1} < \cdots < \lambda_{i+j} \approx \lambda_{i+j+1}$$

Possible reasons:

1. The eigenvalue is a multiple (at least double) eigenvalue; e.g. check $\angle(x_{i+j}, x_{i+j+1})$
2. An eigenvalue in $[\lambda_i, \lambda_{i+j}]$ has been previously missed out;
3. A spurious eigenvalue arose in the interval $[\lambda_i, \lambda_{i+j}]$.

In cases 2,3 the local number of λ_{i+j} is raised by 1, and λ_{i+j} is accepted as an $(i+j+1)$th eigenvalue.

In both cases we determine the additional eigenvalue θ ("suspect") and its local number $\ell + m$ and we expand the search space

$$\hat{\mathcal{V}} = \text{span}\{\mathcal{V}, MT(\theta)x_\theta\}$$

aiming at the corresponding Ritz pair (θ, x_θ).

Workaround

In the course of the computation it may happen that the algorithm converges to an eigenvalue twice

$$\lambda_i < \lambda_{i+1} < \cdots < \lambda_{i+j} \approx \lambda_{i+j+1}$$

Possible reasons:

1. The eigenvalue is a multiple (at least double) eigenvalue; e.g. check $\angle(x_{i+j}, x_{i+j+1})$
2. An eigenvalue in $[\lambda_i, \lambda_{i+j}]$ has been previously missed out;
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By the minmax principle all eigenvalues of the projected problem $T_{\hat{V}}(\lambda)\hat{y} = 0$ are less than or equal to the corresponding ones of $T_V(\lambda)y = 0$.

Either of the following happens:

- $T_{\hat{V}}(\cdot)$ has exactly $j+1$ eigenvalues in $[\lambda_i, \lambda_{i+j}]$ i.e. the additional eigenvalue has left the interval of interest;
- There still are $j+2$ eigenvalues $\lambda_i, \ldots, \lambda_{i+j}, \hat{\theta} \in [\lambda_i, \lambda_{i+j}]$, and it holds $\hat{\theta} \leq \theta$.
  - $\|T(\hat{\theta})\hat{V}x_{\hat{\theta}}\|$ is small $\Rightarrow$ the additional eigenvalue converged. Then we adjust the enumeration and continue the iterative method;
  - Otherwise we repeat the expansion of the subspace until the sequence of additional eigenvalues has been moved out of the interval $[\lambda_i, \lambda_{i+j}]$ or has converged to a previously missed out regular eigenvalue.

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If more than one additional eigenvalue exist in $[\lambda_i, \lambda_{i+j}]$ after we detected a replicate eigenvalue they all can be treated in the way one after the other.

We identify the suspect eigenvalue by assigning the numbers of the eigenvalues in $\lambda_i, \ldots, \lambda_{i+j}$ of $T(\lambda)x = 0$ within the actual search subspace $\mathcal{V}$. The not matched eigenvalues of the projected problem $T_V(\lambda)x = 0$ are suspect.

In practice the search subspace $\mathcal{V}$ usually contains approximations rather than the exact eigenvectors. Thus while identifying the nonlinear eigenvalue λ_i we assign the number of the eigenvalue $\mu(\lambda_i)$ of the linear problem $T_V(\lambda_i)$ with minimal absolute value.

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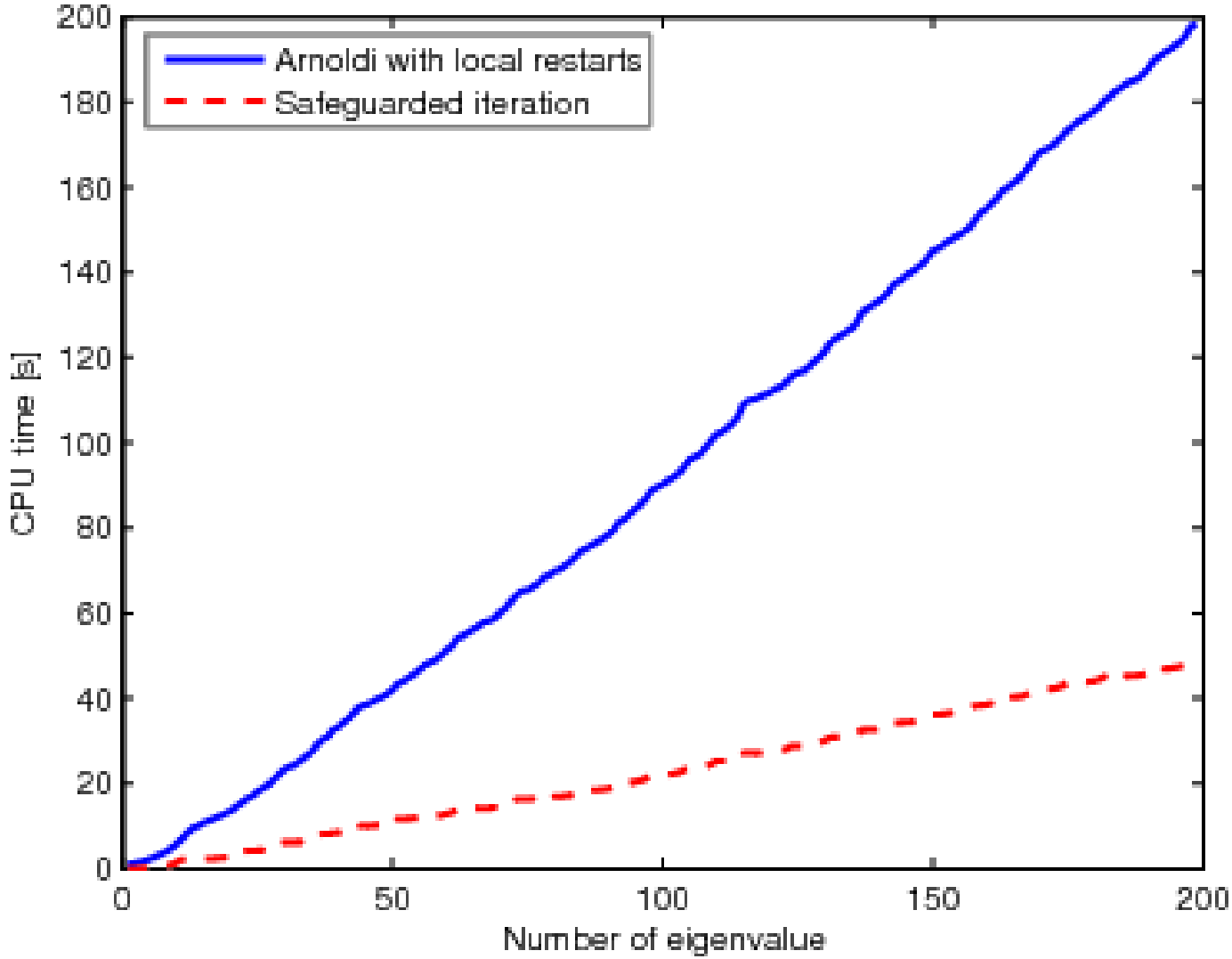
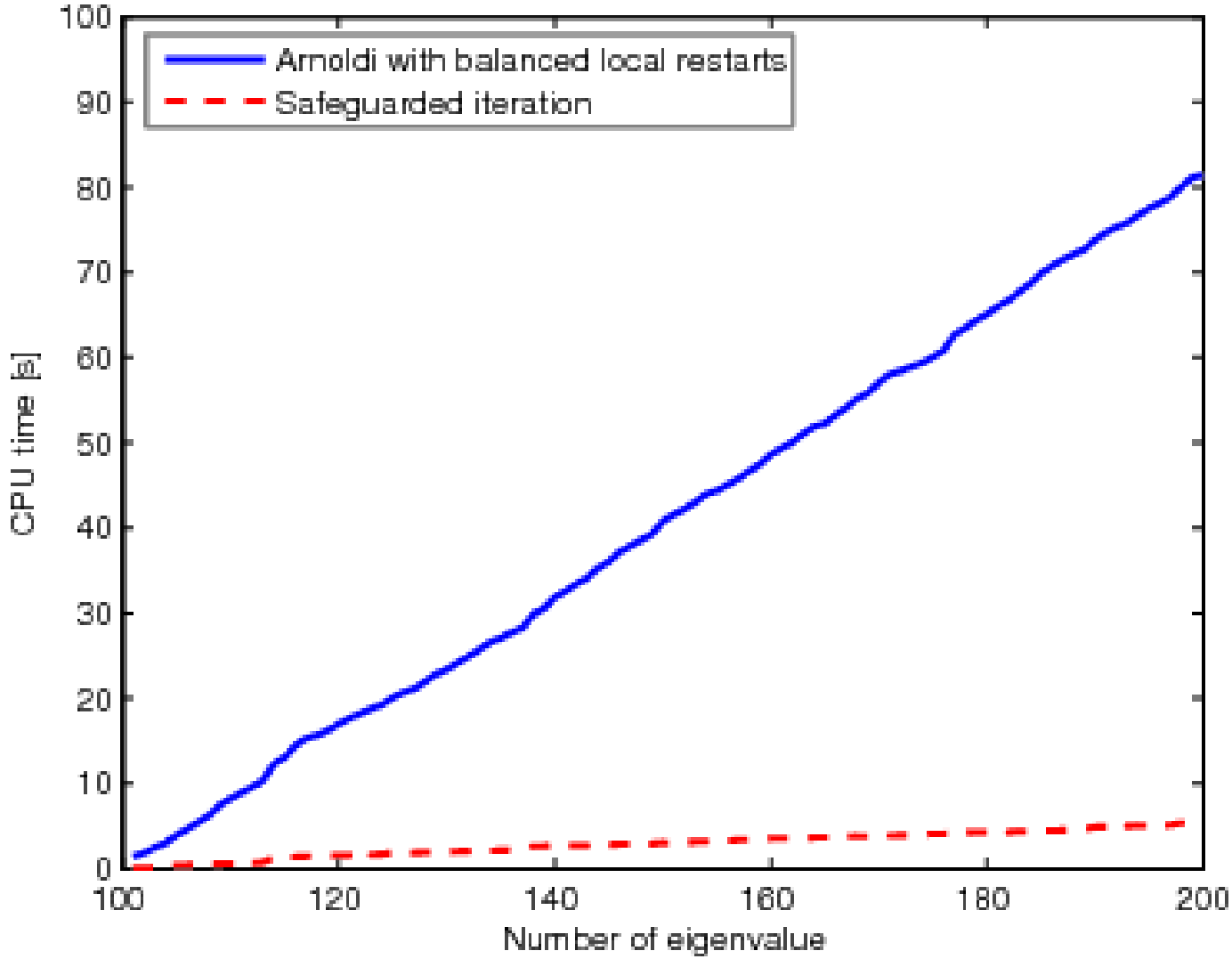
Nonlinear Arnoldi with local restarts

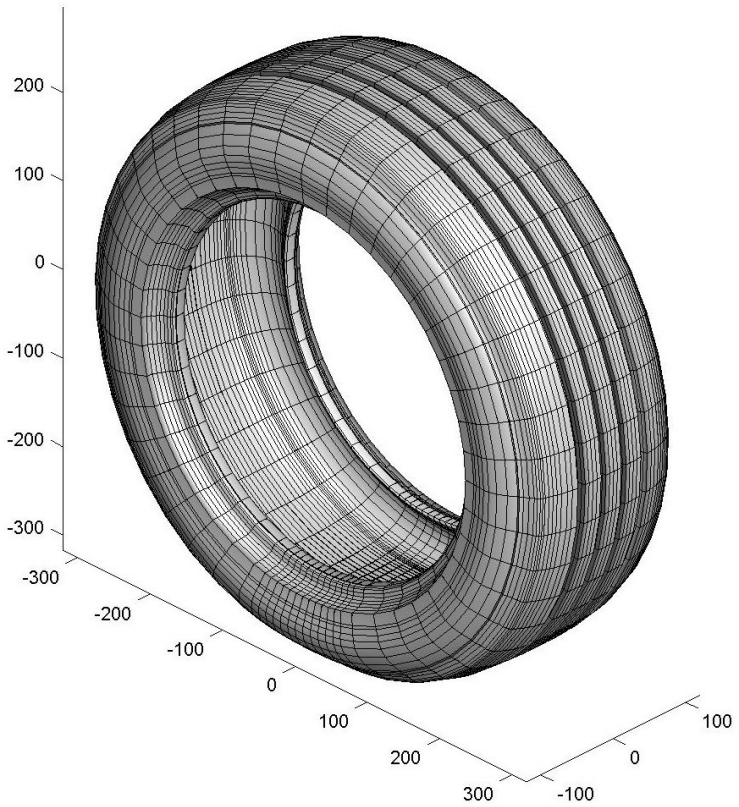


Nonlinear Arnoldi with balanced local restarts



205/55R16-91H tire, Continental AG

19 different materials are used in the finite element model. The model includes the stress approximating the air pressure in the tire. The tire is pressed on the track and is rotating at a rate corresponding to a vehicle speed of 50 km/h.



Balanced vs. non-balanced restarts

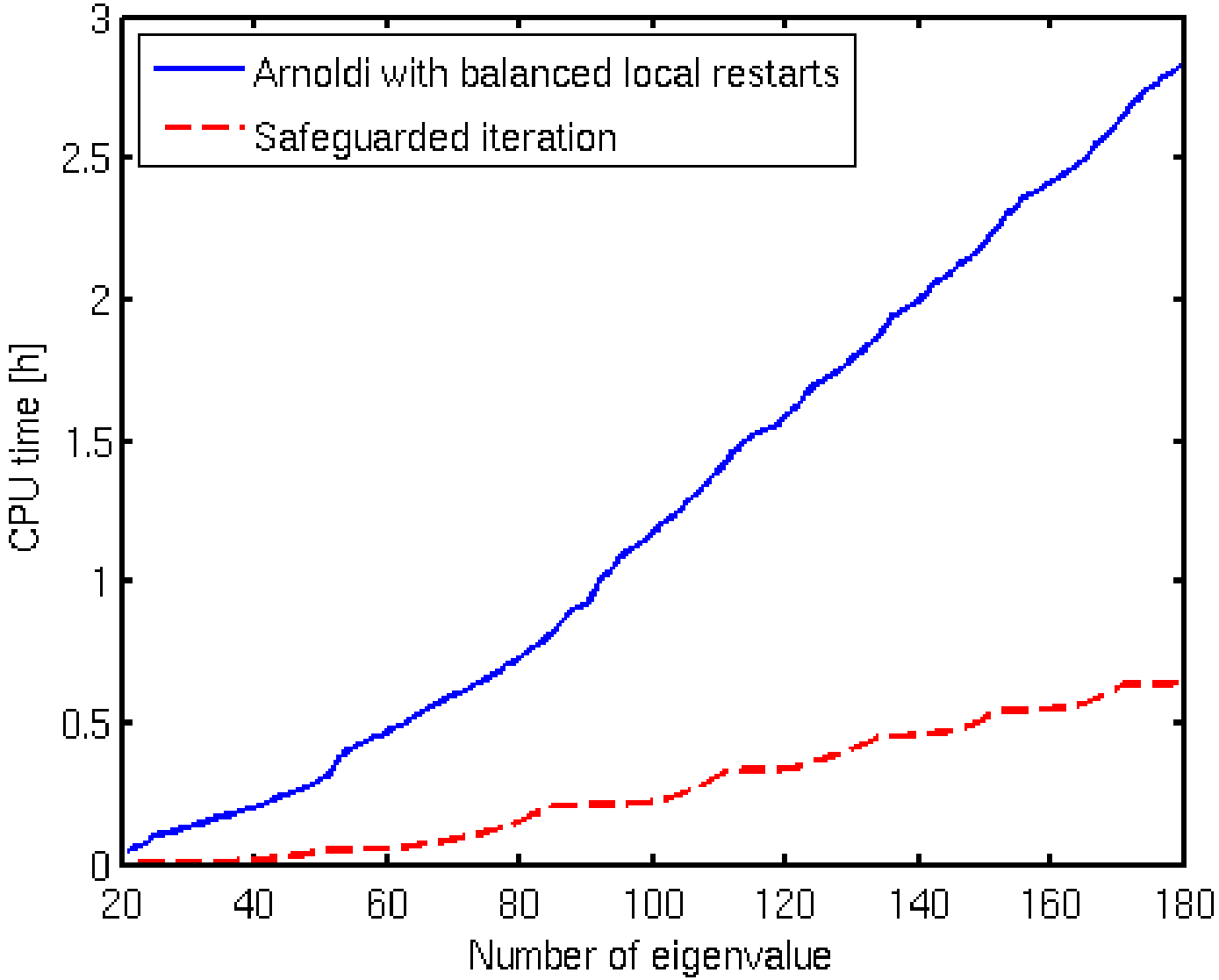
Tests run on a single processor of SGI Altix 4700 machine with 96 Itanium 2 Madison 9M processors (1.6GHz, 6MB L3, single core) sharing 320 GB RAM.

We implemented the algorithm in Fortran 90 using the standard linear algebra packages BLAS and LAPACK from SCSL hardware optimized library and SPARSKIT2. The LU decompositions and subsequent system solves are computed with the PARDISO routines from the Intel MKL library.

CPU times for the computation of all eigenfrequencies in the range $[475, 2000]$Hz without and with balanced restarts.

	MAX_DIM	CPU time [h]	# LU
N.Arn.	230	4.02	5
Balanced N.Arn.	230	2.83	8
N.Arn.	200	3.46	6
Balanced N.Arn.	200	2.84	8

NA with balanced restarts



Conclusions

- We have a local restart technique for nonlinear eigenvalue problems admitting minmax characterization which effectively prevents the superlinear growth with the eigenvalue number
- It can be used within different iterative methods
- It can be used to compute eigenvalues in the interior of the spectrum if it can be initialized
- We developed strategies for dealing with spurious eigenvalues
- The approach can be generalized to rational eigenvalue problems where the numbering is not unique on the real axis.