Trimmed linearizations of (structured) matrix polynomials

Volker Mehrmann

TU Berlin
Institut für Mathematik

DFG Research Center
MATHEON
Math. for Key Technologies

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Higher order DAEs and nonlinear eigenvalue problems

Nonlinear eigenvalue problems are usually motivated by the solution of linear higher order differential equations.

$$\sum_{i=0}^{k} A_i x^{(i)} = f(t).$$

The dynamics of this DAE can be analyzed via the spectral properties of the polynomial eigenvalue problem

$$(\sum_{i=0}^{k} A_i \lambda^i) x = 0.$$

First order fomulations, ‘Linearization’

Introduce new variables

$$y^T = \left[\ y_1, y_2, \ldots, y_k\ \right]^T = \left[\ x, \dot{x}, \ldots, x^{(k-1)}\ \right]^T$$

to turn the high order system into first order system

$$E\dot{y} + Ay = g$$

with the associated linear matrix polynomial

$$L(\lambda) = \lambda E + A.$$

Linearization

Definition: For a $n \times n$ matrix polynomial $P(\lambda)$, a matrix pencil $L(\lambda) = \lambda E + A$ of size $nk \times nk$ is called linearization of $P(\lambda)$, if there exist nonsingular unimodular matrices (i.e., of constant nonzero determinant) $S(\lambda)$, $T(\lambda)$ such that

$$S(\lambda)L(\lambda)T(\lambda) = \mathrm{diag}(P(\lambda), I_{(k-1)n}).$$

Difficulties with this definition of linearization

- Linearization preserves the algebraic and geometric multiplicities of all finite eigenvalues.
- Classical linearization produces unnecessary long eigenvector/principal vector chains for eigenvalue ∞.
- There are difficulties in the proper representation of the singular part.
- Linearization theory for matrix polynomials does not match the theory for DAEs.

Do we need to transform to first order?

Pros

- Simpler analysis for first order systems and linear ev. problems.
- Not good methods for matrix polynomials.
- No generalization of Kronecker form for matrix polynomials.

Cons

- The solvability of the equation may be destroyed for DAEs, M./Shi 2006.
- The condition number (sensitivity) may increase. De Boor/Kreiss pardoxon 1986.
- Standard intergration numerical methods may fail. Sand 2002.
- The size of the problem is increased and symmetry structures may be lost.

First order form in Linear Algebra

The Euler-Lagrange equations of a linear constrained and damped mechanical system have the form

$$\begin{array}{rcl} M\ddot{x} + D\dot{x} + Kx + G^T\mu & = & f(t) \\ Gx & = & 0. \end{array}$$

Here M, D, K are mass, damping and stiffness matrices, G describes the constraint, f a forcing function and μ the Lagrange multiplier. The associated matrix polynomial is

$$P(\lambda) = \lambda^2 \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K & G^T \\ G & 0 \end{bmatrix}.$$

If M is positive definite and G has full row rank, then $P(\lambda)$ has chains associated with the eigenvalue ∞ of length 4.

First order form in DAE theory

In multibody dynamics one introduces the new variable $y = \dot{x}$ and **not a variable** $\gamma = \dot{\mu}$. This gives the first order system

$$\begin{array}{rcl} M\dot{y} + Dy + Kx + G^T\mu & = & f(t), \\ \dot{x} & = & y, \\ Gx & = & 0 \end{array}$$

or the associated linear matrix pencil

$$\tilde{L}(\lambda) = \lambda \begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} D & K & G^T \\ -I & 0 & 0 \\ 0 & G & 0 \end{bmatrix},$$

which has a chain associated with ∞ of length 3.

Different goals in Linear Algebra/DAE theory

- In the eigenvalue problem we want to know eigenvalues, eigenvectors, deflating subspaces, the length of chains associated with eigenvalues, and canonical forms.
- In the differential equation we want the solution, study the dynamics of the system, analyze stability, resonances, etc.
- The motivation for the study of eigenvalue problems arises from differential equations.
- The theories and definitions should match.

First index reduction, then spectral analysis

In numerical methods for first order DAEs, chains longer than 1 associated to ∞ or singular parts lead to numerical difficulties.

- The length of infinite and singular chains determine the smoothness requirements for the inhomogeneity f.
- Using linear combinations of derivatives of equations (unimodular transformations), reformulate the DAE as $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ where all the eigenvalues ∞ are simple, and the right singular parts do not have chains either. (**Index reduction, strangeness-free formulation**). See book: Kunkel/M. 2006
- One solves the alternative system (with the same solution).
- One analyzes the spectral properties of $\hat{E}\dot{x} = \hat{A}x + \hat{f}$.

First normal form then index reduction

Alternatively we can first compute normal forms.

- Transform to Kronecker/Weierstrass or staircase form first.
- Perform index reduction and reformulate via the normal form.
- Solve the transformed system analytically or numerically.
- Analyze the spectral properties via the normal form.

Index reduction and computation of spectral properties (associated with the finite spectrum) commute.
For first order DAEs both procedures are equivalent.

First order vs higher order

- For order larger than 1 we do not have a canonical form.
- Different first order formulations have different smoothness requirements.
- Companion first order form (linearization) may have stronger smoothness requirements for inhomogeneity than necessary, since unnecessary derivatives of variables are inroduced.
- The proper treatment of singular blocks is unclear.
- **Linearization and index reduction do not commute since they may lead to different smoothness requirements.**
- Under small perturbations the systems may behave very different.

A simple example

Consider the DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\mu} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Index reduction (inserting the derivatives of the second equation into the first) gives the first order DAE

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} f_1 - f_2 - \dot{f}_2 - \ddot{f}_2 \\ f_2 \end{bmatrix}.$$

This is first order, no first order formulation is necessary.

The associated matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 + \lambda + 1 & 1 \\ 1 & 0 \end{bmatrix}$$

has only the eigenvalue ∞. Using a unimodular transformation from the left with

$$Q(\lambda) = \begin{bmatrix} 1 & -(\lambda^2 + \lambda + 1) \\ 0 & 1 \end{bmatrix}$$

we obtain the first order

$$T(\lambda) = Q(\lambda)P(\lambda) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has only degree 0.

Is this a polynomial of degree 2, or 1 with leading coefficients 0.

Suggested trimmed linearization procedure

- Compute (structured) normal form or staircase form associated with the eigenvalue ∞ and the singular part for high order DAE (matrix polynomial).
- Reduce index (chains) associated with these parts via linear combinations of derivatives of some equations (unimodular transformations) until the infinite eigenvalues are all simple and the singular blocks have no chains
- Perform order reduction (linearization) on the resulting system (matrix polynomial.)

This avoids unnecessary chains associated with ∞ and singular parts and therefore also unnecessary smoothness requirements.

Linearization

New Definition: For a matrix polynomial $P(\lambda)$, a matrix pencil $L(\lambda) = \lambda E + A$ of size $\ell \times \ell$ is called linearization of the $n \times n$ matrix polynomial $P(\lambda)$, if there exist nonsingular unimodular matrices (i.e., of constant nonzero determinant) $S(\lambda), T(\lambda)$ such that

$$S(\lambda)L(\lambda)T(\lambda) = \mathrm{diag}(P(\lambda), I_{\ell-n}).$$

Theorem M./Shi 2006

Consider a linear DAE

$$A_l x^{(l)} + A_{l-1} x^{(l-1)} + \cdots + A_0 x = f(t)$$

The system is equivalent to a system

$$\tilde{x}_1^{(l)} + \sum_{i=0}^{l-1} \sum_{j=i}^{l-1} A_{1,l-j}^{\langle i \rangle} \tilde{x}_{l-j}^{(i)} + \sum_{i=0}^{l-1} A_{1,l+2}^{\langle i \rangle} \tilde{x}_{l+2}^{(i)} = \tilde{f}_1(t),$$

$$\tilde{x}_2^{(l-2)} + \sum_{i=0}^{l-2} \sum_{j=i}^{l-2} A_{2,l-1-j}^{\langle i \rangle} \tilde{x}_{l-1-j}^{(i)} + \sum_{i=0}^{l-2} \left(A_{2,1}^{\langle i \rangle} \tilde{x}_1^{(i)} + A_{2,l+2}^{\langle i \rangle} \tilde{x}_{l+2}^{(i)} \right) = \tilde{f}_2(t),$$

$$\vdots$$

$$\tilde{x}_{l+1}^{(0)} = \tilde{f}_{l+1}(t),$$

$$0 = \tilde{f}_{l+2}.$$

Properties of this system

- Partial normal (Kronecker/Smith) form associated with ∞ and right singular chains.
- Existence, uniqueness of solutions, consistency of initial conditions, and minimal smoothness requirements for f (perturbation index μ) can be read off.
- All right singular chains and chains at ∞ have length 0.
- First order form without introducing unnecessary variables.
- Solutions are in one-to-one correspondence, since no linear combinations with derivatices are used from the right.
- Computation of this form is not numerically stable.
- A derivative array approach gives analogous results and can be implemented numerically.
- Structure is not preserved, we would rather like a staircase form.

Theorem Byers/M./Xu 2007 Staircase form.

Let $A_i \in \mathbb{C}^{m,n}$ $i = 0, \ldots, k$. Then, the tuple $(A_k, \ldots, A_0)$ is unitarily equivalent to a matrix tuple $(\hat{A}_k, \ldots, \hat{A}_0) = (UA_kV, \ldots, UA_0V)$, where all terms $\hat{A}_i$, $i = 0, \ldots, k$, have the form

$$
\left[\begin{array}{cccc|c|cccc}
A & A & A & \ldots & \ldots & \ldots & A & A & A_l^{(i)} \\
A & A & A & \ldots & \ldots & \ldots & \iddots & A_{l-1}^{(i)} & 0 \\
A & A & A & \ldots & \ldots & \iddots & \iddots & 0 & 0 \\
\vdots & \vdots & \vdots & \iddots & \iddots & A_1^{(i)} & \iddots & \vdots & \vdots \\
\hline
\vdots & \vdots & \iddots & \iddots & A_0^{(i)} & 0 & \ldots & \vdots & \vdots \\
\hline
\vdots & \iddots & \iddots & \tilde{A}_1^{(i)} & \iddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots & \vdots & \vdots \\
A & \tilde{A}_{l-1}^{(i)} & 0 & \iddots & \iddots & \vdots & \vdots & \vdots & 0 \\
\tilde{A}_l^{(i)} & 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0
\end{array}\right],
$$

Properties of this staircase form

- Each of the blocks $A_j^{(i)}$ $i = 0, \dots, k,\ j = 1, \dots, l$ either has the form $\begin{bmatrix} \Sigma & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \end{bmatrix}$,
- Each of the blocks $\tilde{A}_j^{(i)}$ $i = 1, \dots, k,\ j = 1, \dots, l$ either has the form $\begin{bmatrix} \Sigma \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- For each j only of the $A_j^{(i)}$ and $\tilde{A}_j^{(i)}$ is nonzero.
- In the tuple of middle blocks $(A_0^{(k)}, \dots, A_0^{(k)})$ (essentially) no k of the coefficients have a common nullspace.
- In structured case we use congruence to preserve the structure.
- If we include also index reduction then we can proceed further.

Theorem Byers/M./Xu 07 The staircase procedure combined with index reduction will end up with a tuple of middle blocks $(A_0^{(k)}, \dots, A_0^{(k)})$ which has a growing anti-triangular block-structure

$$\left(\begin{bmatrix} \Sigma_k & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} A_{11}^{(k-1)} & A_{12}^{(k-1)} & 0 & \dots & 0 \\ A_{21}^{(k-1)} & \Sigma_{k-1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}, \dots, \right.$$

$$\left. \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & \dots & A_{1,k-1}^{(1)} & 0 \\ \vdots & \ddots & \dots & A_{2,k-1}^{(1)} & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ A_{k-1,1}^{(1)} & A_{k-1,2}^{(1)} & \dots & \Sigma_1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} & \dots & \dots & A_{1,k}^{(0)} \\ \vdots & \ddots & \dots & \dots & A_{2,k}^{(0)} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ A_{k-1,1}^{(0)} & A_{k-1,2}^{(0)} & \dots & A_{k-1,k-1}^{(0)} & A_{k-1,k}^{(0)} \\ A_{k,1}^{(0)} & A_{k,2}^{(0)} & \dots & A_{k,k-1}^{(0)} & \Sigma_0 \end{bmatrix} \right)$$

associated with a regular matrix polynomial that has only simply eigenvalues associated with ∞.

Consequences

- Singular parts can be deflated.
- All the long chains associated with ∞ can be deflated.
- We obtain trimmed linearizations.
- Reversing the order of the middle block, we can do the same for eigenvalue 0.
- The transformation can be done in a structure preserving way.
- Structure preserving trimmed linearizations can be obtained.

Conclusions and future work.

- DAEs and nonlinear eigenvalue problems are important in many applications.
- The mathematical treatment does not match.
- Index reduction and linearization do not commute.
- New trimmed linearization techniques are available.
- Structured staircase forms have been derived.
- Deflation of eigenvalues and singular blocks directly in nonlinear problem is possible.
- Code implementation is still to be done.

Thank you very much for your attention.

information, papers, codes etc

http://www.math.tu-berlin.de/~mehrmann