Frequency isolation algorithms for hyperbolic quadratic eigenvalue problems

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Outline

Hyperbolic QEPs

- Consider **quadratic e-value problem (QEP)**

$$Q(\lambda)u = (\lambda^2 M + \lambda C + K)u = 0.$$

with M, C, K symmetric, positive definite $n \times n$.

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- Nice properties of hyperbolic QEPs:
  - $2n$ real and semisimple e-values
  - e-values can be obtained by bisection

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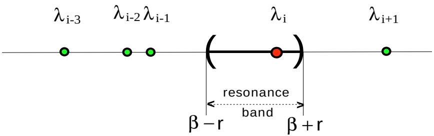

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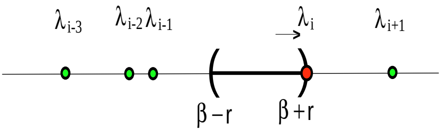

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Frequency isolation problem: Given a resonance band $I_r = (\beta - r, \beta + r)$ and a vibrational system (M, C, K) with some eigenvalue in $(\beta - r, \beta + r)$, **redesign the system** in such a way that the new system (M^*, C^*, K^*)

- has **no eigenvalue in the resonance band**, and
- **is close to** (M, C, K) in some sense.

Frequency isolation algorithms: the undamped case

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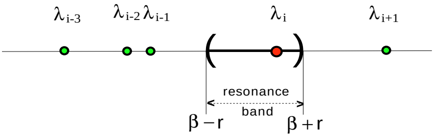

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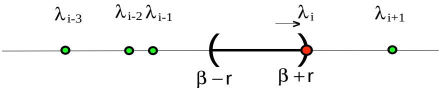

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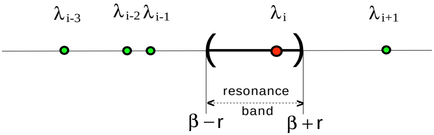

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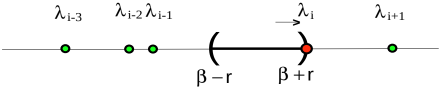

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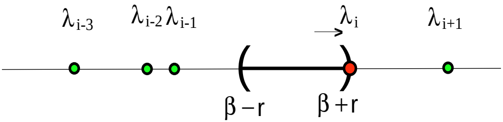
Very expensive: cost well over $O(n^4)$ + quite restrictive conditions.

How to improve it?

Fixing a target spectrum is unnatural, since “good" eigenvalues are not allowed to change at all



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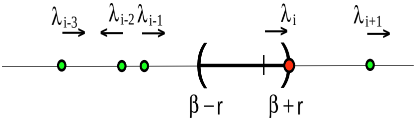

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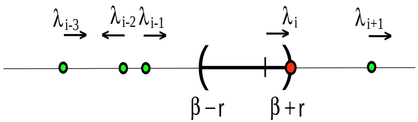

Idea of new algorithm: Identify a direction in (M, C, K) space along which:

- variation of “bad" eigenvalues is **maximal**, and
- variation of “good" eigenvalues is **minimal**.

Then, modify (M, C, K) along this direction up to isolation.

Outline

A simple situation

As an example, consider hyperbolic QEP

$$Q(\lambda) = \lambda^2 M + \lambda C + K$$

with

$$M = \mathrm{diag}(m) = \mathrm{diag}(m_1, \ldots, m_n), \qquad m_i > 0,$$
$$C = \mathrm{diag}(c) = \mathrm{diag}(c_1, \ldots, c_n), \qquad c_i > 0,$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}, \qquad k_i > 0$$

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$\lambda_j = \lambda_j(m,c,k) \quad$ for $\quad (m,c,k) = (m_1, \ldots, m_n, c_1, \ldots, c_n, k_1, \ldots, k_n) \in \mathbb{R}^{3n}.$

Thus,

Work in parameter space (m,c,k) instead of in matrix space (M,C,K)

How to choose directions in (m, c, k) space?

- Consider initial configuration $(m, c, k) = (m_1, \ldots, m_n, c_1, \ldots, c_n, k_1, \ldots, k_n) \in \mathbb{R}^{3n}$, and call

  good e-values those s.t. $\lambda_j(m, c, k) \notin (\beta - r, \beta + r)$

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$$\lambda_j((m,c,k)+\Delta) = \lambda_j(m,c,k) + < \nabla \lambda_j(m,c,k), \Delta > + \dots$$

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  1) as large as possible for bad λ_j

  2) as small as possible for good λ_j $\rightarrow$ zero for good λ_j **i.e.,**

$$\Delta \perp \nabla \lambda_j(m, c, k) \quad \text{for all} \quad \text{good } \lambda_j.$$

The basic isolation algorithm

Since

$$\Delta = (\delta m, \delta c, \delta k) \perp \nabla \lambda_j(m,c,k) \qquad \text{for all} \quad \text{good} \ \ \lambda_j,$$

denote

$$V_{good}^{\perp} = \{ w \in \mathbb{R}^{3n} : < \nabla \lambda_j(m,c,k), w >= 0 \quad \text{for all} \ \ \text{good} \ \ \lambda_j \}.$$

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2) Given $w_{\max}$ from **Stage 1)**, find smallest $\alpha^* \in \mathbb{R}$ s.t. e-values corresp. to

$$(m^*, c^*, k^*) = (m,c,k) + \alpha^* w_{\max}$$

are all outside the resonance band.

Implementation of Stage 1): finding w_{max}

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- $w_{\max} \equiv$ singular vector corresp. to $\sigma_{\max}$ of scalar product matrix

OVERALL COST: $O(n^3)$

Implementation of Stage 2): detecting isolation

Stage 2) Given optimal direction $w_{\max} \in V_{good}^{\perp}$, find smallest $\alpha^* \in \mathbb{R}$ such that spectrum of

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QEP is hyperbolic $\longrightarrow$ can use bisection on α to find how many e-values of

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COST: $O(n)$ per bisection step

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Higham, Tisseur & Van Dooren '02

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All these constraints lead to a maximal range for α

$$-\tau^- \leq \alpha \leq \tau^+$$

for appropriate thresholds $\tau^-, \tau^+ > 0$.

Algorithm only works if there are no e-values in $(\beta - r, \beta + r)$ either for $\alpha = -\tau^-$ or for $\alpha = \tau^+$ $\longrightarrow$ provides starting interval for bisection.

The basic isolation algorithm: Shortcomings

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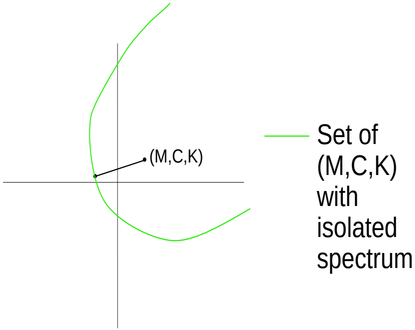

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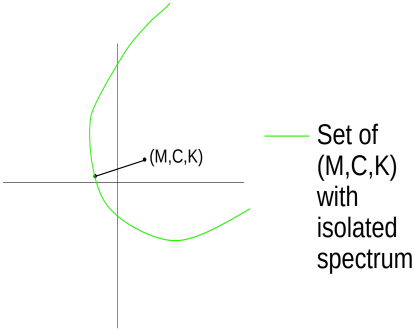

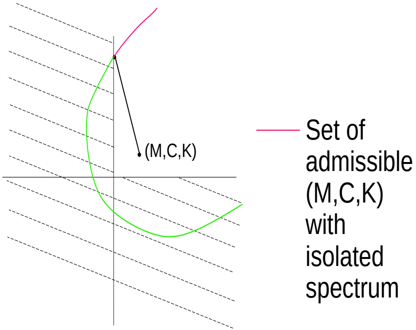

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Both difficulties may be overcome

A continuation algorithm:

Instead of trying to isolate in one single run, repeat basic isolation procedure over and over, setting $(M_0, C_0, K_0) = (M, C, K)$ and updating

$$(M_{i+1}, C_{i+1}, K_{i+1}) = (M_i, C_i, K_i) + h_i\, w_{\max}^{(i)},$$

with some appropriate, small step size h_i, where $w_{\max}^{(i)}$ is the optimal direction at step i. Graphically,

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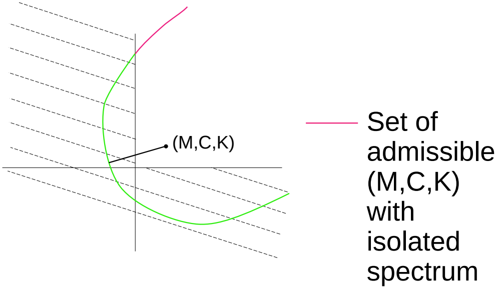

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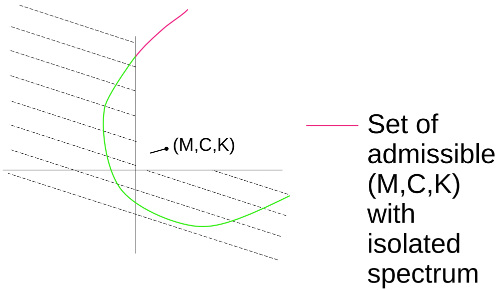

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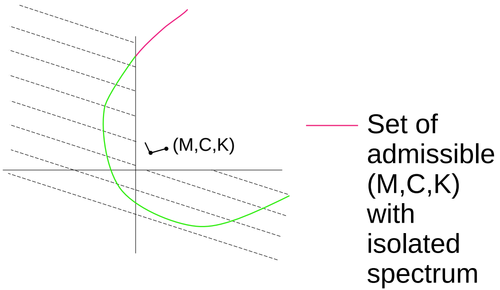

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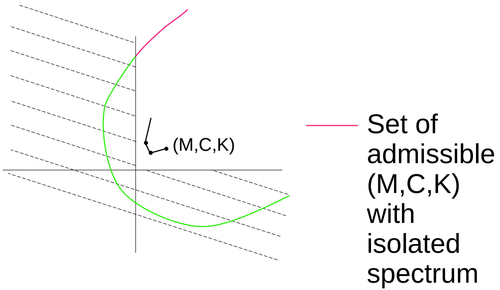

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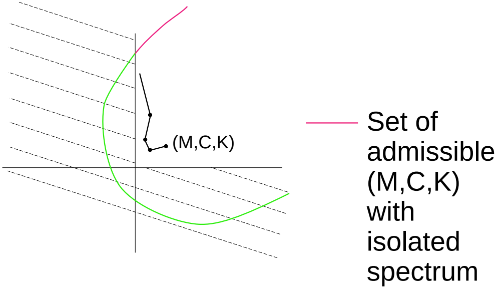

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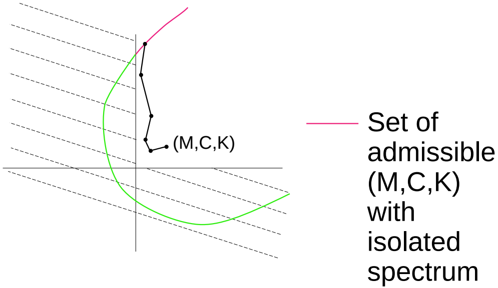

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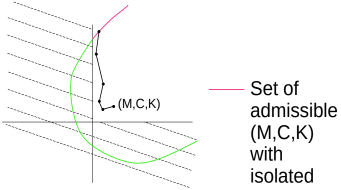

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$$(M_{i+1}, K_{i+1}) = (M_i, K_i) + h_i\, w_{\max}^{(i)}, \qquad (M_0, K_0) = (M, K)$$

Several possible choices for h_i. Best performance so far:

Greedy version: Try to isolate at each step. If not possible, then advance as far as possible in the optimal direction and repeat.

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Several possible choices for h_i. Best performance so far:

Greedy version: Try to isolate at each step. If not possible, then advance as far as possible in the optimal direction and repeat.

Compute thresholds τ_i^+, τ_i^- and optimal direction $w_{\max}^{(i)}$ at step i.

- If possible, compute α_i^* isolating the spectrum, take $h_i = \alpha_i^*$ and stop.
- If not, take h_i equal either to τ_i^+ or $-\tau_i^-$ and continue.

Outline

Numerical experiments: comparison with EKS

- Both basic and continuation algorithm implemented for tridiagonal undamped case ($C = 0$) in FORTRAN POWER STATION 4.0 on a PC with IEEE arithmetic
- Generate 1000 random 10×10 QEPs with one single e-value in the resonance band. Radius r of resonance band inversely proportional to $\delta < 1$:

  the smaller the parameter δ, the smaller the radius r

- Compare obtained solutions with solutions provided by 'Box' algorithm by Egaña, Kuhl & Santos '02 (EKS) whenever possible:
  - Box algorithm works only for low dimension (too expensive).
  - Box algorithm works only for uniform distributions of initial e-values.

Numerical experiments: Box vs. basic isolation

Generate 1000 random 10×10 mass-spring systems for each value of δ.

Set

- $m^*_{box}, k^*_{box} \equiv$ solutions computed by the 'Box' algorithm of Egaña, et al.
- m^*_{β}, k^*_{β}: solutions computed by the basic isolation algorithm.

$$Dist^{(\square)}_{m,k} = \frac{\|(m,k) - (m^*,k^*)_{\square}\|}{\|(m,k)\|},$$

where $\square$ is either 'box´ or 'β'. Define the quotients

$$Q^{(\beta)}_{m,k} = \frac{Dist^{(box)}_{m,k}}{Dist^{(\beta)}_{m,k}}, \qquad Q^{(\beta)}_{T} = \frac{\text{CPU time for Box alg.}}{\text{CPU time for basic isolation alg.}}$$

and the percentage $p_{m,k}$ of cases with $Q^{(\beta)}_{m,k} < 1$.

Numerical experiments: Box vs. basic isolation

Generate 1000 random 10×10 mass-spring systems for each value of δ.

$$Dist_{m,k}^{(\square)} = \frac{\|(m,k) - (m^*,k^*)_{\square}\|}{\|(m,k)\|},$$

where $\square$ is either 'box' or 'β'. Define the quotients

$$Q_{m,k}^{(\beta)} = \frac{Dist_{m,k}^{(box)}}{Dist_{m,k}^{(\beta)}}, \qquad Q_T^{(\beta)} = \frac{\text{CPU time for Box alg.}}{\text{CPU time for basic isolation alg.}}$$

and the percentage $p_{m,k}$ of cases with $Q_{m,k}^{(\beta)} < 1$.

	$Q_{m,k}^{(\beta)}$			$Q_T^{(\beta)}$		
δ	Average	Min	$p_{m,k}$	Average	Min	isolated
0.1	0.74	0.09	53.1%	514	61	97.8%
0.15	0.9	0.1	66.4%	420	53	96.9%
0.2	1.31	0.15	82.2%	359	51	89.1%
0.4	2.38	0.39	97.3%	228	48	59.6%

Numerical experiments: Box vs. continuation

Generate 1000 random 10×10 mass-spring systems for each value of δ.

Set

- $m^*_{box}, k^*_{box} \equiv$ solutions computed by the 'Box' algorithm of Egaña, et al.
- m^*_{cont}, k^*_{cont}: solutions computed by the continuation algorithm (greedy version).

$$Dist^{(\square)}_{m,k} = \frac{\|(m,k) - (m^*,k^*)_{\square}\|}{\|(m,k)\|},$$

where $\square$ is either 'box´ or '*cont*'. Define the quotients

$$Q^{(cont)}_{m,k} = \frac{Dist^{(box)}_{m,k}}{Dist^{(cont)}_{m,k}}, \qquad Q^{(cont)}_{T} = \frac{\text{CPU time for Box alg.}}{\text{CPU time for continuation alg.}}$$

and the percentage $p_{m,k}$ of cases with $Q^{(cont)}_{m,k} < 1$.

Numerical experiments: Box vs. continuation

Generate 1000 random 10×10 mass-spring systems for each value of δ.

$$Dist_{m,k}^{(\square)} = \frac{\|(m,k) - (m^*,k^*)_{\square}\|}{\|(m,k)\|},$$

where $\square$ is either 'box' or '*cont*'. Define the quotients

$$Q_{m,k}^{(cont)} = \frac{Dist_{m,k}^{(box)}}{Dist_{m,k}^{(cont)}}, \qquad Q_T^{(cont)} = \frac{\text{CPU time for Box alg.}}{\text{CPU time for continuation alg.}}$$

and the percentage $p_{m,k}$ of cases with $Q_{m,k}^{(cont)} < 1$.

	$Q_{m,k}^{(cont)}$			$Q_T^{(cont)}$	
δ	Average	Min	$p_{m,k}$	Average	Min
0.2	0.66	0.04	61.2%	532	67
0.3	0.67	0.009	65.3%	398	65
0.4	0.83	0.016	69.3%	276	50
0.6	1.06	0.04	77.4%	218	36

Summary:

New, directional algorithm for the frequency isolation problem proposed for hyperbolic QEPs.

Tested only for tridiagonal undamped case.

- **Basic algorithm**: $O(n^3)$, but isolation guaranteed only for systems close to non-resonance.
- **Continuation algorithm**:
  - Cost $O(n^3)$ per step.
  - Much faster than 'Box' algorithm.
  - Same quality of approximations as Box alg.
  - More robust: works irrespective of spectral distribution or distance to non-resonance.