Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients

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June 27th, 2011
Overview

1. Stochastic differential equations (SDEs)

2. Convergence for SDEs with globally Lipschitz continuous coefficients

3. Convergence for SDEs with superlinearly growing coefficients
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3. Convergence for SDEs with superlinearly growing coefficients
Consider

- $d, m \in \mathbb{N}$, $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$,
- an $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion $W : [0, T] \times \Omega \to \mathbb{R}^m$,
- continuous functions $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ and
- an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable $\xi : \Omega \to \mathbb{R}$ with $\mathbb{E}\|\xi\|^p < \infty \forall p \in [1, \infty)$.

Let $X : [0, T] \times \Omega \to \mathbb{R}$ be an up to modifications unique adapted stochastic process with continuous sample paths satisfying

$$X_t = \xi + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Short form:

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = \xi, \quad t \in [0, T].$$
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- \( d, m \in \mathbb{N} \), \( T \in (0, \infty) \) and a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a normal filtration \((\mathcal{F}_t)_{t \in [0, T]}\),
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with \( X_0 = \xi \) and \( t \in [0, T] \).

- The goal of this talk is to solve (1).
- Since explicit solutions are typically not available, we want to solve (1) approximatively: Computational Stochastics.
- Problem (1) is not contained in the standard literature in computational stochastics, e.g.,
  - Kloeden & Platen (1992) and
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since \( \mu \) and \( \sigma \) are not assumed to be globally Lipschitz continuous.
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Examples of SDEs

**Black-Scholes model**; \( \bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty) \):

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\begin{align*}
dX_t &= \bar{\mu} X_t \, dt + \bar{\sigma} X_t \, dW_t, \\
x_0 &= x_0, \quad t \in [0, T].
\end{align*}
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**Lewis stochastic volatility model**; \( \bar{\mu}, \hat{\mu}, \tilde{\mu}, \tilde{\sigma} \in (0, \infty) \) appropriate:

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\begin{align*}
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with \( X_0 = x_0 \in (0, \infty)^2 \) and \( t \in [0, T] \).

**An SDE with a cubic drift and additive noise**:

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\begin{align*}
dX_t &= -X_t^3 \, dt + dW_t, \\
x_0 &= 0, \quad t \in [0, 1].
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    dX_t^{(1)} &= \bar{\mu} X_t^{(1)} \, dt + \left( X_t^{(2)} \right)^{\frac{1}{2}} X_t^{(1)} \, dW_t^{(1)} \\
    dX_t^{(2)} &= X_t^{(2)} \left( \hat{\mu} - \tilde{\mu} X_t^{(2)} \right) \, dt + \bar{\sigma} \left( X_t^{(2)} \right)^{\frac{3}{2}} \, dW_t^{(2)}
\end{align*}
$$

with $X_0 = x_0 \in (0, \infty)^2$ and $t \in [0, T]$.

**An SDE with a cubic drift and additive noise:**

$$dX_t = -X_t^3 \, dt + dW_t, \quad X_0 = 0, \quad t \in [0, 1].$$
Examples of SDEs

**Black-Scholes model;** $\tilde{\mu}, \tilde{\sigma}, x_0 \in (0, \infty)$:

\[
dX_t = \tilde{\mu} X_t \; dt + \tilde{\sigma} X_t \; dW_t, \quad X_0 = x_0, \quad t \in [0, T].
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\end{align*}
\]

\[
\begin{align*}
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Overview

1. Stochastic differential equations (SDEs)

2. Convergence for SDEs with globally Lipschitz continuous coefficients

3. Convergence for SDEs with superlinearly growing coefficients
The **explicit Euler scheme** $Y_n^N : \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N = \xi$ and

$$Y_{n+1}^N = Y_n^N + \frac{T}{N} \cdot \mu(Y_n^N) + \sigma(Y_n^N) \left( W_{(n+1)T}^N - W_{nT}^N \right)$$

for all $n \in \{0, 1, \ldots, N - 1\}$, $N \in \mathbb{N}$.

**Theorem (Maruyama 1955; Kloeden and Platen 1992)**

Let $\mu$ and $\sigma$ be globally Lipschitz continuous. Then there exists a real number $C \in [0, \infty)$ such that

$$\left( \mathbb{E} \left[ \| X_T - Y_N^N \|^2 \right] \right)^{\frac{1}{2}} \leq C \cdot N^{-\frac{1}{2}}$$

for all $N \in \mathbb{N}$.

Convergence of Euler’s method is well understood in the Lipschitz case.
The **explicit Euler scheme** $Y_n^N : \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N = \xi$ and

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$$

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Let $\mu$ and $\sigma$ be globally Lipschitz continuous. Then there exists a real number $C \in [0, \infty)$ such that

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Stochastic differential equations (SDEs)

Convergence for SDEs with globally Lipschitz continuous coefficients
Convergence for SDEs with superlinearly growing coefficients

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Let $\mu$ and $\sigma$ be globally Lipschitz continuous. Then there exists a real number $C \in [0, \infty)$ such that

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Overview

1. Stochastic differential equations (SDEs)

2. Convergence for SDEs with globally Lipschitz continuous coefficients

3. Convergence for SDEs with superlinearly growing coefficients
Convergence of Euler’s method

\[ \lim_{N \to \infty} \mathbb{E} \left[ \left\| X_T - Y_N^T \right\|^2 \right] = 0, \quad \lim_{N \to \infty} \mathbb{E} \left[ \left\| X_T \right\|^2 \right] - \mathbb{E} \left[ \left\| Y_N^T \right\|^2 \right] = 0 \]

for SDEs with superlinearly growing coefficients such as

an SDE with a cubic drift and additive noise:

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remained an open problem.
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Compare: This divergence is in fundamental contrast to the convergence of the explicit Euler method in the deterministic case: In the case $\xi$ deterministic and $\sigma(x) = 0$ for all $x \in \mathbb{R}$, the explicit Euler method does converge: $\lim_{N \to \infty} \mathbb{E}\left[|X_T - Y_N^N|^2\right] = 0$.

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Convergence for SDEs with superlinearly growing coefficients

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Examples of SDEs

Divergence of Euler’s method

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holds for:

A SDE with a cubic drift and additive noise:

\[
dX_t = -X_t^3 \, dt + dW_t, \quad X_0 = 0, \quad t \in [0, 1].
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Variance process in the Lewis stochastic volatility model:

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Some ideas in the divergence proof of Euler’s method

Fix large $N \in \mathbb{N}$ and consider

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The well known instability of Euler’s method then gives

$$Y_0^N = N \quad \text{positive},$$

$$Y_1^N = Y_0^N - \frac{1}{N} (Y_0^N)^3 = N - N^2 \approx -N^2 \quad \text{negative},$$

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and, in particular, $|Y_N^N| \geq N^{2^N}$ (at least double exponential growth in $N$).
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\begin{align*}
    dX_t &= -X_t^3 \, dt, \\
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    t &\in [0, 1].
\end{align*}
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Now consider the SDE

$$dX_t = -X_t^3 \, dt + dW_t, \quad X_0 = 0, \quad t \in [0, 1]$$

and define “events of instabilities”

$$\Omega_N := \left\{ \omega \in \Omega : \sup_{1 \leq k \leq N-1} \left| W_{k+1}^N(\omega) - W_k^N(\omega) \right| \leq 1, \ W_1^N(\omega) - W_0(\omega) \geq 3N \right\}$$

for all $N \in \mathbb{N}$. Estimates on the previous slide then indicate that

$$|Y_N^N(\omega)| \geq N^{2(N-1)} \quad (2)$$

for all $\omega \in \Omega_N$, $N \in \mathbb{N}$. Moreover,

$$e^{-cN^3} \leq \mathbb{P}[\Omega_N] \leq e^{-\tilde{c}N^3} \quad (3)$$

for all $N \in \mathbb{N}$ with $c, \tilde{c} \in (0, \infty)$ appropriate. Combining (2) and (3) shows

$$\mathbb{E} \left[ |Y_N^N| \right] \geq \mathbb{P}[\Omega_N] \cdot N^{2(N-1)} \geq e^{-cN^3} \cdot N^{2(N-1)} \xrightarrow{N \to \infty} \infty.$$

This gives

$$\lim_{N \to \infty} \mathbb{E} \left[ |X_T - Y_N^N|^2 \right] = \infty, \quad \lim_{N \to \infty} \mathbb{E} \left[ |X_T|^2 \right] - \mathbb{E} \left[ |Y_N^N|^2 \right] = \infty.$$
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Now consider the SDE

\[ dX_t = -X_t^3 \, dt + dW_t, \quad X_0 = 0, \quad t \in [0, 1] \]

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\[ \Omega_N := \left\{ \omega \in \Omega : \sup_{1 \leq k \leq N-1} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \ W_{\frac{1}{N}}(\omega) - W_0(\omega) \geq 3N \right\} \]

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Implicitness is a way to overcome this problem

Let $\mu$ be globally one-sided Lipschitz continuous, i.e., there exists a real number $c \in [0, \infty)$ such that

$$\langle x - y, \mu(x) - \mu(y) \rangle \leq c \|x - y\|^2$$

for all $x, y \in \mathbb{R}^d$.

The implicit Euler scheme $\tilde{Y}_N^n : \Omega \to \mathbb{R}, n \in \{0, 1, \ldots, N\}$, is given by $\tilde{Y}_0^N = \xi$ and

$$\tilde{Y}_{n+1}^N = \tilde{Y}_n^N + \frac{T}{N} \cdot \mu(\tilde{Y}_{n+1}^N) + \sigma(\tilde{Y}_n^N) \left( W_{(n+1)T}^N - W_{nT}^N \right)$$

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Convergence of the implicit Euler scheme

Theorem (Higham, Mao & Stuart 2002)

Let $\sigma$ be globally Lipschitz continuous and let $\mu$ be globally one-sided Lipschitz continuous with an at most polynomially growing continuous derivative. Then there exists a real number $C \in [0, \infty)$ such that

$$\left( \mathbb{E} \left[ \| X_T - \tilde{Y}_N^N \|^2 \right] \right)^{\frac{1}{2}} \leq C \cdot N^{-\frac{1}{2}}$$

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Stochastic differential equations (SDEs)

Convergence for SDEs with globally Lipschitz continuous coefficients

Convergence for SDEs with superlinearly growing coefficients

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The **implicit Euler scheme** requires **additional computational effort** for computing the zero of a nonlinear equation in each time step.
Nonlinear SDEs: The implicit Euler method converges but simulations for it on a computer require additional computational effort. The explicit Euler method is explicit and easy to simulate but

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left| X_T - Y^N_T \right|^2 \right] = \infty
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(4)

Higham (2010) reviews this divergence and a long time divergence result and states “... it is clear that any other explicit numerical method can suffer the same fate. This brings us to a key point. Unlike in the deterministic ODE case, for non-linear SDEs, we introduce implicitness not in the hope of improving efficiency by allowing larger stepsize, but in the hope of obtaining a method that satisfies the fundamental requirements of accuracy and stability.” This motivated us to ask:

Is there any explicit numerical method which does not suffer from (4)?

Is there any explicit and easily simulatable numerical method which converges strongly for SDEs with superlinearly growing coefficients?

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**Theorem (Hutzenthaler, J & Kloeden 2010)**

Let $\sigma$ be globally Lipschitz continuous and let $\mu$ be **globally one-sided** Lipschitz continuous with an at most polynomially growing continuous derivative. Then there exists a real number $C \in [0, \infty)$ such that

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Arnulf Jentzen

Nonlinear SDEs
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\left( \mathbb{E} \left[ \| X_T - \tilde{Y}_N^N \|^2 \right] \right)^{\frac{1}{2}} \leq C \cdot N^{-\frac{1}{2}}
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**Theorem (Hutzenthaler, J & Kloeden 2010)**

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Some ideas in the convergence proof of the tamed Euler scheme

In the case $\sigma(x) = 0$ for all $x \in \mathbb{R}$ we have

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\| \bar{Y}_{n+1}^N \| = \| \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) \| \leq \| \bar{Y}_n^N \| + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \| \leq \| \bar{Y}_n^N \| + 1
$$

for all $n \in \{0, 1, \ldots, N - 1\}$ and therefore $\| \bar{Y}_N^N \| \leq \| \xi \| + N$ for all $N \in \mathbb{N}$ (at most linear growth in $N$).

Recall $|Y_N^N| \geq N^{2^N}$ (at least double exponential growth in $N$) for the ODE

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Recall $|Y_n^N| \asymp N^{(2N)}$ (at least double exponential growth in $N$) for the ODE

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\| \bar{Y}_{n+1}^N \| \leq \| \bar{Y}_n^N \| + \frac{T_N \cdot \mu(\bar{Y}_n^N)}{1 + \frac{T_N}{N} \cdot \| \mu(\bar{Y}_n^N) \|} \leq \| \bar{Y}_n^N \| + 1
$$

for all $n \in \{0, 1, \ldots, N - 1\}$ and therefore $\| \bar{Y}_N^N \| \leq \| \xi \| + N$ for all $N \in \mathbb{N}$ (at most linear growth in $N$).

Recall $|Y_N^n| \asymp N^{2^N}$ (at least double exponential growth in $N$) for the ODE

$$
dX_t = -X_t^3 \, dt, \quad X_0 = \xi = N, \quad t \in [0, 1].$$
Some ideas in the convergence proof of the tamed Euler scheme

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\leq \left\| \bar{Y}_n^N \right\| + 1
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dX_t = -X_t^3 \, dt, \quad X_0 = \xi = N, \quad t \in [0, 1].$$
In the case $\sigma(x) = 0$ for all $x \in \mathbb{R}$ we have

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\]
In the case \( \sigma(x) = 0 \) for all \( x \in \mathbb{R} \) we have

\[
\| \bar{Y}_{n+1}^N \| = \left\| \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) \frac{1}{1 + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \|} \right\| \leq \| \bar{Y}_n^N \| + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \| \frac{1}{1 + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \|} \\
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$$

$$
\leq \left\| \vec{Y}_n^N \right\| + 1
$$

for all $n \in \{0, 1, \ldots, N - 1\}$ and therefore $\| \vec{Y}_N^N \| \leq \| \xi \| + N$ for all $N \in \mathbb{N}$ (at most linear growth in $N$).

Recall $|Y_N^N| \approx N^{2N}$ (at least double exponential growth in $N$) for the ODE

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dX_t = -X_t^3 \, dt, \quad X_0 = \xi = N, \quad t \in [0, 1].$$
In the case $\sigma(x) = 0$ for all $x \in \mathbb{R}$ we have

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\| \bar{Y}_{n+1}^N \| = \| \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) \|_{1 + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \|} \leq \| \bar{Y}_n^N \| + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \| \\
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Recall $|Y_N^N| \asymp N^{(2^N)}$ (at least double exponential growth in $N$) for the ODE

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\frac{dX_t}{dt} = -X_t^3, \quad X_0 = \xi = N, \quad t \in [0, 1].
$$
The **tamed Euler method** may still behave badly on appropriate events of instabilities! However, on such events it behaves (at most linear growth in \( N \)) not as bad as the explicit Euler method (at least double exponential growth in \( N \)). This and some other arguments yield

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{0 \leq n \leq N} \| \bar{Y}^N_n \|^p \right] < \infty
\]  

for all \( p \in [1, \infty) \). Moreover, note that

\[
\bar{Y}_{n+1}^N = \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) + \sigma(\bar{Y}_n^N) \left( W_{(n+1)\frac{T}{N}} - W_{n\frac{T}{N}} \right) - \left( \frac{T}{N} \right)^2 \frac{\mu(\bar{Y}_n^N) \cdot \| \mu(\bar{Y}_n^N) \|}{1 + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \|}
\]  

for all \( n \in \{0, 1, \ldots, N - 1\}, N \in \mathbb{N} \), i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields

\[
\lim_{N \to \infty} \mathbb{E} \left[ \| X_T - \bar{Y}^N_T \|^2 \right] = 0.
\]
The **tamed Euler method** may still behave badly on appropriate events of instabilities! However, on such events it behaves (at most linear growth in $N$) not as bad as the explicit Euler method (at least double exponential growth in $N$). This and some other arguments yield

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$$

for all $p \in [1, \infty)$. Moreover, note that

$$
\bar{Y}^N_{n+1} = \bar{Y}^N_n + \frac{T}{N} \cdot \mu(\bar{Y}^N_n) + \sigma(\bar{Y}^N_n) \left( W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \right) - \left( \frac{T}{N} \right)^2 \frac{\mu(\bar{Y}^N_n) \cdot \left\| \mu(\bar{Y}^N_n) \right\|}{1 + \frac{T}{N} \cdot \left\| \mu(\bar{Y}^N_n) \right\|} \quad (6)
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for all $n \in \{0, 1, \ldots, N - 1\}$, $N \in \mathbb{N}$, i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields

$$
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$$
The **tamed Euler method** may still behave badly on appropriate **events of instabilities**! **However**, on such events it behaves (at most linear growth in \(N\)) not as bad as the explicit Euler method (at least double exponential growth in \(N\)). This and some other arguments yield

\[
\sup_{N \in \mathbb{N}} \mathbb{E}\left[ \sup_{0 \leq n \leq N} \| \tilde{Y}_n^N \|^p \right] < \infty
\]  

(5)

for all \(p \in [1, \infty)\). Moreover, note that

\[
\tilde{Y}_{n+1}^N = \tilde{Y}_n^N + \frac{T}{N} \cdot \mu(\tilde{Y}_n^N) + \sigma(\tilde{Y}_n^N) \left( W_{(n+1)T/N} - W_{nT/N} \right) - \left( \frac{T}{N} \right)^2 \frac{\mu(\tilde{Y}_n^N) \cdot \| \mu(\tilde{Y}_n^N) \|}{1 + \left( \frac{T}{N} \right) \cdot \| \mu(\tilde{Y}_n^N) \|} 
\]

(6)

for all \(n \in \{0, 1, \ldots, N - 1\}, N \in \mathbb{N}\), i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields \(\lim_{N \to \infty} \mathbb{E}\left[ \| X_T - \tilde{Y}_N^N \|^2 \right] = 0\).
The **tamed Euler method** may still behave badly on appropriate events of instabilities! However, on such events it behaves (at most linear growth in $N$) not as bad as the explicit Euler method (at least double exponential growth in $N$). This and some other arguments yield

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{0 \leq n \leq N} \left\| \tilde{Y}_n^N \right\|^p \right] < \infty \quad (5)$$

for all $p \in [1, \infty)$. Moreover, note that

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for all $n \in \{0, 1, \ldots, N - 1\}$, $N \in \mathbb{N}$, i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields $\lim_{N \to \infty} \mathbb{E}[\| X_T - \tilde{Y}_N^N \|^2] = 0$. Arnulf Jentzen

**Nonlinear SDEs**
The **tamed Euler method** may still behave badly on appropriate events of instabilities! However, on such events it behaves (at most linear growth in \( N \)) not as bad as the **explicit Euler method** (at least double exponential growth in \( N \)). This and some other arguments yield

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for all \( p \in [1, \infty) \). Moreover, note that

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\tilde{Y}_{n+1}^N = \tilde{Y}_n^N + \frac{T}{N} \cdot \mu(\tilde{Y}_n^N) + \sigma(\tilde{Y}_n^N) \left( W_{(n+1)T}^N - W_{nT}^N \right) \\
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\]  

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for all \( n \in \{0, 1, \ldots, N - 1\} \), \( N \in \mathbb{N} \), i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields

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The **tamed Euler method** may still behave badly on appropriate **events of instabilities**! However, on such events it behaves (at most linear growth in $N$) not as bad as the explicit Euler method (at least double exponential growth in $N$). This and some other arguments yield

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{0 \leq n \leq N} \| \bar{Y}_n^N \|^p \right] < \infty$$

(5)

for all $p \in [1, \infty)$. Moreover, note that

$$\bar{Y}_{n+1}^N = \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) + \sigma(\bar{Y}_n^N) \left( W_{(n+1)T} - W_{nT} \right)$$

$$- \left( \frac{T}{N} \right)^2 \frac{\mu(\bar{Y}_n^N) \cdot \| \mu(\bar{Y}_n^N) \|}{1 + \frac{T}{N} \cdot \| \mu(\bar{Y}_n^N) \|}$$

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for all $p \in [1, \infty)$. Moreover, note that

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Arnulf Jentzen

Nonlinear SDEs
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**Stochastic differential equations (SDEs)**

**Convergence for SDEs with globally Lipschitz continuous coefficients**

**Convergence for SDEs with superlinearly growing coefficients**

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Arnulf Jentzen  Nonlinear SDEs
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Arnulf Jentzen
Nonlinear SDEs
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\bar{Y}_{n+1}^N = \bar{Y}_n^N + \frac{T}{N} \cdot \mu(\bar{Y}_n^N) + \sigma(\bar{Y}_n^N) \left( W_{(n+1)T}^N - W_{nT}^N \right)
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for all \( n \in \{0, 1, \ldots, N - 1\} \), \( N \in \mathbb{N} \), i.e., **tamed Euler method** coincides with **explicit Euler method** up to terms of second order. Using ideas in Higham, Mao & Stuart (2002), (5) and (6) yields

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left\| X_T - \bar{Y}_N^N \right\|^2 \right] = 0.
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The **tamed Euler method** may still behave badly on appropriate **events of instabilities**! However, on such events it behaves (at most linear growth in $N$) not as bad as the **explicit Euler method** (at least double exponential growth in $N$). This and some other arguments yield

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\[ dX_t = -X_t^5 \, dt + X_t \, dW_t, \quad X_0 = 1, \quad t \in [0, 1]. \]
Summary and message of the talk

For the nonlinear SDEs considered here:

- The explicit Euler scheme, does, in general, not converge strongly to the exact solution of the SDE (see Hutzenthaler, J & Kloeden 2009).
- This is in fundamental constrast to the convergence of the explicit Euler method to the exact solution in the deterministic case.
- There exist explicit numerical approximation methods which overcome the lack of convergence of the explicit Euler method and which converge strongly to the exact solution of the SDE (see Hutzenthaler, J & Kloeden 2010). For convergence, there is thus no need of implicitness.
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Many thanks for your attention!