Provability Logic
and
the Arithmetics of a Theory

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Overview

Provability Logic

Solovay’s Theorem

An Example
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Solovay’s Theorem

An Example
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Provability Logic

Solovay’s Theorem

An Example
The Logic GL

The Logic GL is the normal modal logic given by the following principles.

G1 ⊢ φ ⇒ ⊢ □φ
G2 ⊢ □(φ → ψ) → (□φ → □ψ)
G3 ⊢ □φ → □□φ
G4 ⊢ □(□φ → φ) → □φ

G3 follows from the other axioms

The logic GL is complete for finite, transitive, irreflexive Kripke Frames.
We define:

- $\square^0 \perp := \perp$,
- $\square^{n+1} \perp := \square \square^n \perp$,
- $\square^\infty \perp := \top$.

Shavrukov calls these *lies*.

Let $a$ range over $0, 1, \ldots \infty$. $\text{GL}_a$ is $\text{GL} + \square^a \perp$. 
Consider any interpretation $N : S_2^1 \rightarrow U$, where $U$ is $\Delta_1^b$-axiomatized. We call $N$ an *arithmetical interpretation in $U$*. 

We consider the arithmetical formula giving the axioms of a theory $U$ as part of the data for $U$. Source and target theory are part of the data for an interpretation. 

An arithmetical interpretation of GL in $N$ is a mapping $\sigma$ from the formulas of GL to the sentences of $U$, that commutes with the propositional connectives, such that:

\begin{align*}
\Box \phi &:= (\Box u \phi^\sigma)^N := (\text{prov}_U(\Box^\sigma \phi))^N.
\end{align*}
The Provability Logic of a Theory

We define:

- $\phi \in \text{prl}(N)$ iff, for all $N$-translations $\sigma$, $U \vdash \phi^\sigma$,
- $\phi \in \text{prl}_{\text{all}}(U)$ iff, for all arithmetics $N$ in $U$, $\phi \in \text{prl}(N)$.
- $\text{deg}(N) := \min(\{a \mid \square^a \bot \in \text{prl}(N)\})$.
- $\text{deg}_{\text{all}}(U) := \min(\{a \mid \square^a \bot \in \text{prl}_{\text{all}}(U)\})$.

If there are no arithmetics in $U$, then $\text{deg}_{\text{all}}(U) = 0$.

if $N : S_2^1 \to U$ is an identical embedding $E_{S_2^1, U}$, we speak also of the provability logic $\text{prl}(U)$ of $U$.

Theorem
GL is always part of $\text{prl}(N)$, for an arbitrary arithmetic $N$ in any theory $U$. 

Overview

Provability Logic

Solovay’s Theorem

An Example
Solovay’s Theorem for Single Arithmetics

Let $\Gamma$ be a set of arithmetical sentences. Then for $A$ in $\Gamma$, $\vdash A \rightarrow \Box_U A^N$ holds. Here $N$ is an arithmetic in $U$ such that $U \vdash (T_2 + (N, \exists \Pi^b_{\text{sent}})\text{-comp})^N$.

Then $\text{prl}(N) = GL_{\text{deg}(N)}$.

The proof uses the careful analysis of the Solovay argument due to De Jongh, Jumelet and Montagna. The substitution instances are disjunctions of conjunctions of $\forall \Delta^b_1$-sentences and $\exists \Pi^b_1$-sentences.
Other Results in the Same Niche 1

**Theorem**
Suppose $U$ contains a $\Sigma_1^0$-sound arithmetic $N$. Then, there is an arithmetic $M$ in $U$, such that $\text{prl}(M) = \text{GL}$.

By bootstrapping and by the second incompleteness theorem $U$ interprets $T_2^1 + \text{incon}(U)$. So, a fortiori, $U$ interprets

$$W := T_2^1 + \{(K, \exists \Pi^b_1, \text{sent}) \text{-comp} \mid K \text{ is an arithmetic in } U\}.$$  

Since, $U$ contains a $\Sigma_1^0$-sound arithmetic, by results of Per Lindström, we can find a faithful interpretation $M$ of $W$. Since $W$ is a true theory, it follows that $\text{prl}(M) = \text{GL}$. 

Theorem
Suppose $A$ is a finitely axiomatized sequential theory. Then, there is an arithmetic $M$ in $A$, such that $\text{prl}(M) = \text{GL}$.

By results of Harvey Friedman and, independently, Jan Krajíček, the theory $A$ contains a $\Sigma^0_1$-sound arithmetic.
The theory CFL is introduced by A. Cordón-Franco, A. Fernández-Margarit and F. F. Lara-Martín as an axiomatization of the boole($\Sigma_1$)-consequences both of $I\Pi_1^-$ and of EA.

CFL is $I\Delta_0$ plus

$$\vdash \exists x S_0(x) \rightarrow \exists x \exists y (2^x = y \land S_0(x)).$$

where $S_0$ is $\Sigma_1(x)$.

CFL is incomparable with $S_2^1$. If we replace $S_2^1$ by CFL in our definition of arithmetic, we get Solovay’s full theorem.

The theory CFL is locally interpretable in Q but not globally interpretable.
Great Open Problems

► What is the provability logic of $S_2^1$?
► What is the provability logic of $S_2^1 + \exists \Pi^b_1$-comp?
► What is the provability logic of $T_2^1$?
► What is the provability logic of $S_2 = \text{I} \Delta_0 + \Omega_1$?

Verbrugge and Razborov:
If $S_2^1 \vdash \exists \Pi^b_1$-comp, then $\text{NP} \cap \text{co-NP} = \text{P}$.
The Initial Arithmetic Ordering

Consider arithmetics $N$ and $M$ in $U$. We define $N \preceq M$ if there is a $U$-definable and $U$-verifiable initial embedding $F$ of $N$ in $M$.

A theory $U$ is \textit{sequential} if it has good sequence coding.

\textbf{Theorem} (Pudlák-Dedekind)
Suppose $U$ is sequential. Then, for all arithmetics $N$ and $M$ in $U$, there is an arithmetic $K$ in $U$ such that $K \preceq N$ and $K \preceq M$.

\textbf{Theorem} (Visser)
Consider a finite set of $\Sigma_1^0$-sentences $S$, a theory $U$ and an arithmetic $N$ in $U$. Then, there is an arithmetic $M \preceq N$, such that

$$U \vdash (T_1^2 + S\text{-comp})^M.$$
Solovay’s Theorem for All Arithmetics of a Given Theory

**Theorem**
Consider any theory $U$. We have: $\text{prl}_{\text{all}}(U) = \text{GL}_{\text{deg}_{\text{all}}}(U)$. 

The proof uses the previous theorem in combination with the work of De Jongh, Jumelet and Montagna.

The theorem also works when $U$ does not contain any arithmetic.
Overview

Provability Logic

Solovay’s Theorem

An Example
The Example

Suppose $A$ is a finitely axiomatized sequential theory. We consider the theory:

$W := A + \{(\Diamond^N \#^N \perp^N) | N \text{ is an arithmetic in } A\}$.

**WARNING:** sloppy formulation.

We have:

- $\deg_{\text{all}}(W) = \infty$,
- for any arithmetic $N$ in $W$, $\deg(N) < \infty$.
- The predicate logic of $W$ is complete $\Pi^0_2$.
- $A \nvdash W$, $A \not\supseteq_{\text{mod}} W$. 
The Lemma 1

Consider a sequential sentence $A$.

- $N$ is $\Sigma_1^0$-veracious in $A$ iff

\[ S_2^1 \vdash \forall S \in \Sigma_1^0\text{-sent} (\Box_A S^N \rightarrow \Box S_2^1 (\text{con}_{\rho(A)}(A) \rightarrow S)). \]

So $\Sigma_1^0$-veracity is the $S_2^1$-verifiable $\Sigma_1^0$-conservativity of $N$ over $\text{ID}_{S_2^1 + \text{con}_{\rho(A)}(A)}$.

- $N$ is strong in $A$ iff $A \vdash \text{con}_{\rho(A)}^N(A)$.

- $N$ is deep in $A$ iff $N$ is both $\Sigma_1^0$-veracious and strong in $A$.

**Theorem**

Suppose that $A$ is a sequential sentence and $N$ is $\Sigma_1^0$-veracious in $A$. Then,

\[ \text{I} \Delta_0 + \text{supexp} \vdash \forall S \in \Sigma_1^0\text{-sent} ((\text{con}(A) \land \Box_A S^N) \rightarrow \text{true}(S)). \]

Here true is a $\Sigma_1^0$ truth predicate.
The Lemma 2

**Theorem**
Suppose $N$ is a deep arithmetic in $A$. We have:

$$S_1^2 \vdash \forall S \in \Sigma^0_1\text{-sent} (\Box_A S^N \leftrightarrow \Box_{S_2^1}(\text{con}_\rho(A)(A) \rightarrow S)).$$

**Theorem**
Both $\Sigma^0_1$-veracity and strength are downwards closed w.r.t. $\preceq$.

**Theorem**
For every arithmetic $N$ in $A$, there is a deep arithmetic $M$ in $A$ with $M \preceq N$.

**Theorem**
Suppose $N$ is $\Sigma^0_1$-veracious in $A$. We have:

$$S_1^2 \vdash \Box_A \Box_{A^N,n} \bot \leftrightarrow \Box_{S_2^1} \Box_A \bot.$$
Strange but True

Suppose $N$ is a deep arithmetic in GB. Then, (suppressing the von Neumann interpretation):

$$\text{GB} + \text{con}(\text{GB}) \vdash \text{con}(\text{GB} + \text{con}^N(\text{GB})).$$

This is \textit{not} an example of a theory proving its own consistency!
We \textit{do} have:

$$\text{GB} + \text{con}(\text{GB}) \not\vdash \text{con}^N(\text{GB} + \text{con}(\text{GB})).$$

and:

$$\text{GB} + \text{con}^N(\text{GB}) \not\vdash \text{con}^N(\text{GB} + \text{con}^N(\text{GB})).$$