Padé via Lanczos for multiple right-hand sides

Karl Meerbergen

Numerics for Control and Simulation

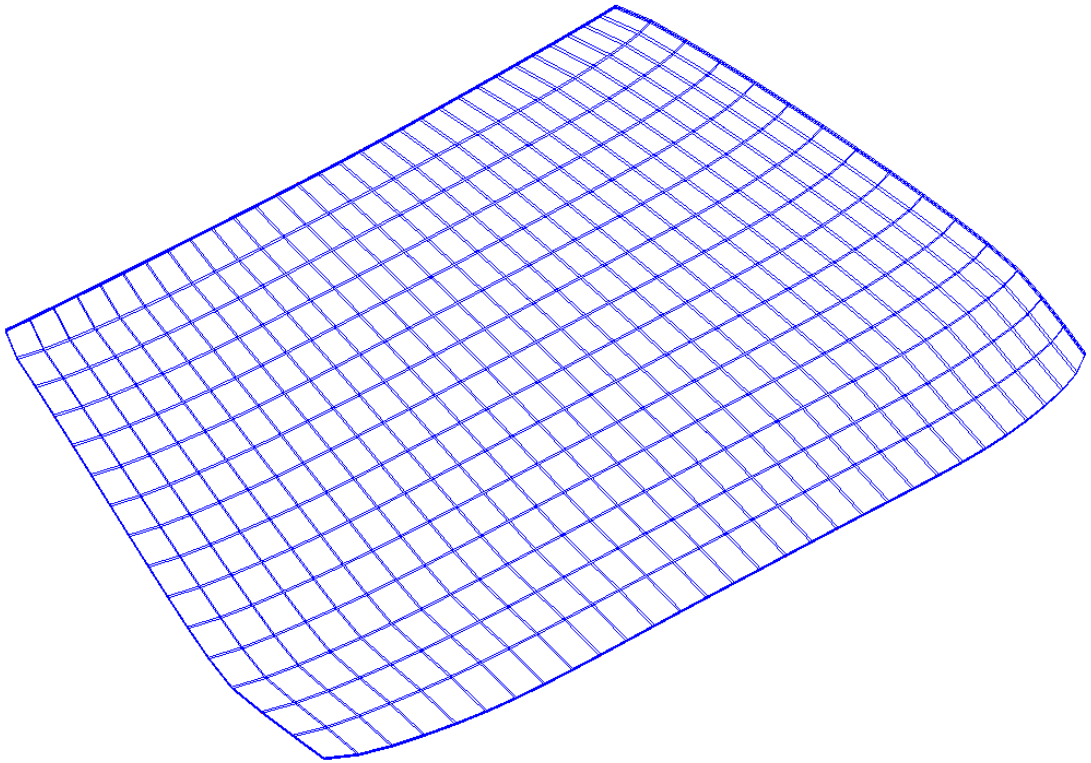
June 17, 2009

Outline

(joint work with Zhaojun Bai)

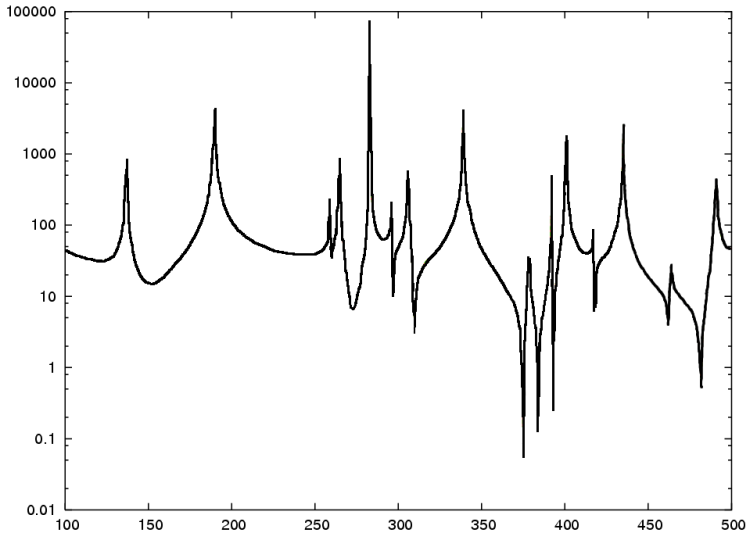
Vibration problems

A car window is subjected to vibrations from outside, including wind. Glass manufacturers want to compute the transmission of noise through windscreens.



Fourier analysis

- Solve
$$(K - \omega^2 M)x = f$$
with K, M : $n \times n$ sparse matrices, symmetric.
- $\omega \in \Omega = [\omega_{\min}, \omega_{\max}]$.
We also define $\Omega^2 = [\omega_{\min}^2, \omega_{\max}^2]$
- x is called the frequency response function.



Alternatives: modal truncation

- Consider the eigendecomposition

$$Ku_j = \lambda_j M u_j$$

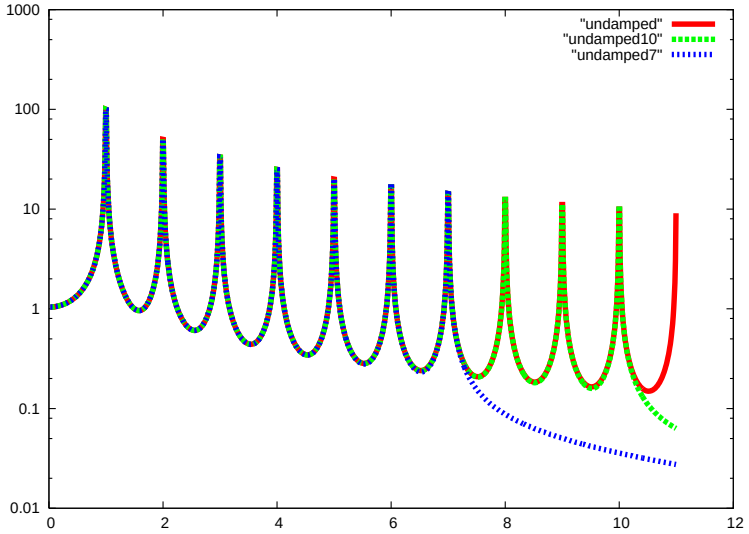
- The solution of $(K - \omega^2 M)x = f$ is

$$x = \sum_{j=1}^{n} u_j \frac{u_j^* f}{\lambda_j - \omega^2}$$

- Rational function with poles λ_j.

Alternative: modal truncation

$$x = \sum_{j=1}^{n} u_j \frac{u_j^* f}{\lambda_j - \omega^2} \approx \sum_{j=1}^{k} u_j \frac{u_j^* f}{\lambda_j - \omega^2}$$



Alternative: AMLS

The solution of

$$(K - \omega^2 M)x = f$$

for $\omega \in \Omega$ is split into two parts.

- Let $U_p = [u_1, \ldots, u_p]$ be the eigenvectors corresponding to the eigenvalues in Ω^2.
- Solve the $p \times p$ problem

$$U_p^*(K - \omega^2 M)U_p z = U_p^* f$$

- Solve the remainder

$$(K - \omega^2 M)y = (I - MU_pU_p^*)f$$

- $x = U_p z + y$

Alternative: AMLS

- Preconditioner for the 'remainder' system :

$$(K - \sigma M)^{-1}$$

- Preconditioned system is

$$Ay = b$$

with

$$\begin{aligned} b &= (I - MU_pU_p^*)(K - \sigma M)^{-1} f \\ A &= (K - \sigma M)^{-1}(K - \omega^2 M) \end{aligned}$$

- In AMLS [Bennighof & Kaplan, 1998], [Ko & Bai, 2007] $K - \sigma M$ is diagonal

Alternative: Lanczos process

- Use a Krylov method for solving $Ay = b$
- Since $x^* MAy = y^* MAx$, we use the Lanczos method with M inner product

- Lanczos method:

1. Let $v_1 = b/\|b\|_M$
2. For $j = 1, \ldots, k$
   - 2.1. Compute Krylov vector $v_{j+1} = Av_j$.
   - 2.2. Orthogonalize v_{j+1} against $v_1, \ldots, v_j$ so that $v_{j+1}^* MV_j = 0$.

- Lanczos vectors $V_k = [v_1, \ldots, v_k]$.
- Recurrence relation: $AV_k - V_k T_k = \beta_k v_{k+1} e_k^*$ with T_k tridiagonal
- $y = V_k T_k^{-1} e_1 \|b\|_M$

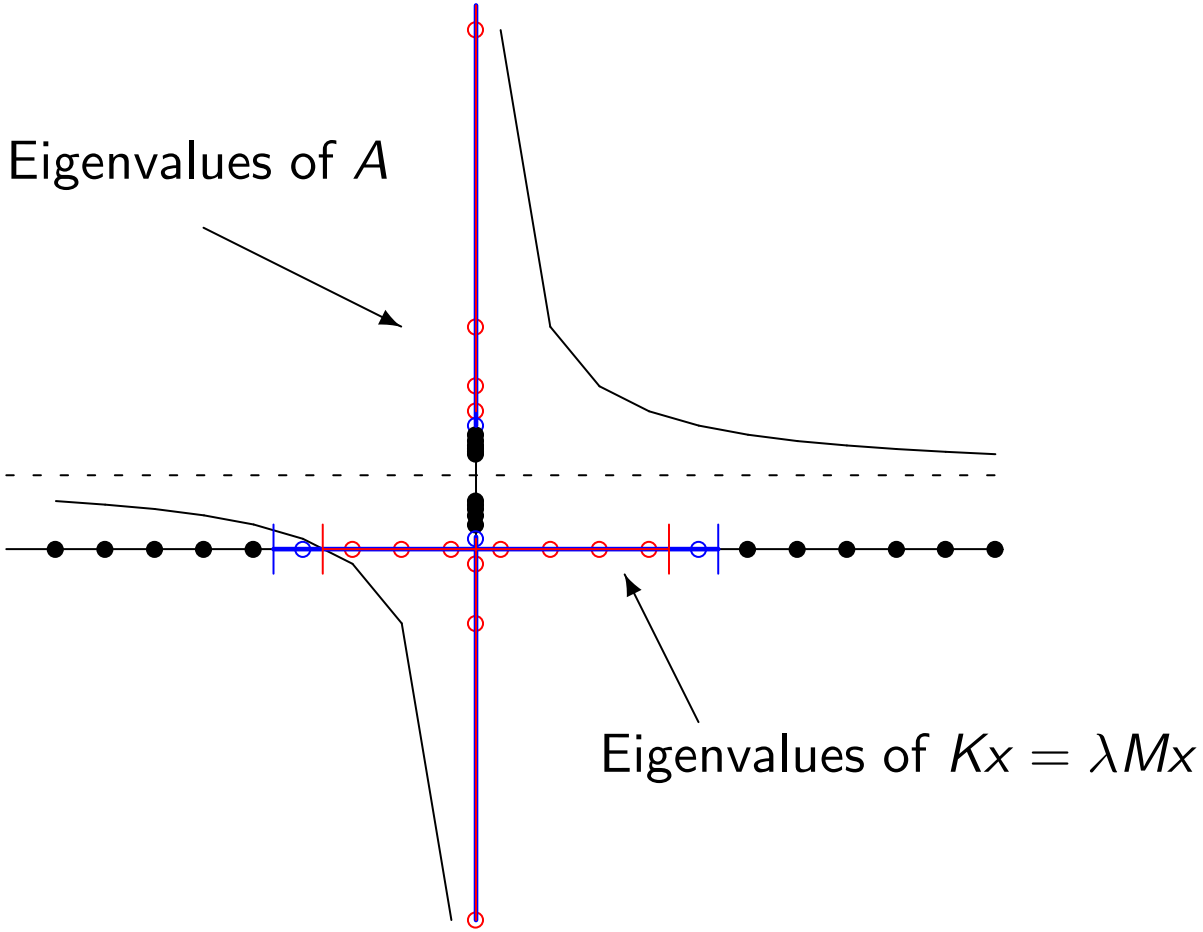
Shifted Lanczos method

- Recall that
$$A = (K - \sigma M)^{-1}(K - \omega^2 M)$$
- $$(K - \sigma M)^{-1}(K - \omega^2 M)V_k - V_k T_k = \beta_k v_{k+1} e_k^*$$
- V_k and T_k can be computed from the shifted Lanczos method.
$$(K - \sigma M)^{-1} M \tilde{V}_k - \tilde{V}_k \tilde{T}_k = \tilde{\beta}_k \tilde{v}_{k+1} e_k^*$$
- $V_k = \tilde{V}_k$ and $T_k = I + (\sigma - \omega^2)\tilde{T}_k$

Shifted linear systems

- Analyzed in the context of model reduction methods
- Feldman, Freund, Bai, Grimme, Sorensen, Van Dooren, Ruhe, Skoogh, Olsson, Simoncini, M., ...
- Connection with eigendecomposition
- Connection with iterative linear solvers
- Connection with rational approximation (Padé)
- Recycling Ritz vectors for parameterized systems: Kilmer & de Sturler 2006

Spectral analysis of deflated iterative method



- Let $B = A|_{\{U_p\}^{\perp_M}}$
- Black eigenvalues only: B is positive definite
- Eigenvalues of B clustered around 1
- spectral radius of $I - B$ is smaller than one

Inexact deflation

- Be careful for deflation with Ritz vectors [Darnell, Morgan, Wilcox 2007]
- Reason is that the system's residual need not be small

$$r(\omega) = R_p z_p + v_{k+1}\zeta_k$$

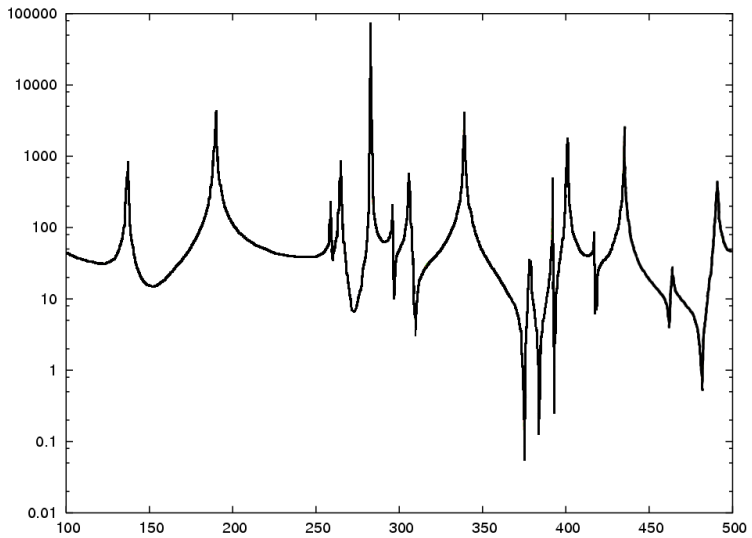
with R_p coming from deflation of Ritz vectors and v_{k+1} the residual term of the iterative method.

Eigenvalue solver

- Use Ritz pairs of the Lanczos method
  - Ritz values: $T_k z = \theta z$
  - Ritz vectors: $\hat{u} = V_k z$
- No exact eigenpairs
- But interesting properties as we now see:

Convergence of Ritz vectors

- Eigenvalues in Ω^2 are computed fairly accurately



- If $\|A\hat{y} - b\|_M \leq \gamma \|y\|_M$ for all $\omega \in \Omega$
- then $\rho_j = \|A\hat{u}_j - \hat{u}_j\hat{\theta}_j\|_M$ with

$$\rho_j \leq \frac{\gamma}{|\lambda_j - \sigma|}$$

Methods

- DRHSL or init-CG
  - Solve $U_p^T(K-\sigma M)^{-1}(K-\omega^2 M)U_p z = U_p^T f$
  - Compute Krylov space for $A = (K-\sigma M)^{-1}M$ and $(I - U_p U_p^T M)(K-\sigma M)^{-1} f$
  - Solve tridiagonal system
  - Add the two components of the solution
- DML
  - Solve $U_p^T(K-\sigma M)^{-1}(K-\omega^2 M)U_p z = U_p^T f$
  - Compute Krylov space for $B = (I - U_p U_p^T M)(K-\sigma M)^{-1}M(I - U_p U_p^T M)$ and $(I - U_p U_p^T M)(K-\sigma M)^{-1} f$
  - Project A on $[U_p \ V_k]$
- In exact arithmetic both methods do the same thing
- DML is more expensive in operations and memory
- but may be more reliable since deflation is imposed explicitly

Padé via Lanczos

- Lanczos method:

$$\hat{x} = \sum_{j=1}^{k} \hat{u}_j \frac{\hat{u}_j^* f}{\hat{\lambda}_j - \omega^2}$$

  - First k moments of $\hat{x}$ and x match.

- With exact deflation:

$$\hat{x} = \sum_{j=1}^{p} u_j \frac{u_j^* f}{\lambda_j - \omega^2} + \sum_{j=p+1}^{k} \hat{u}_j \frac{\hat{u}_j^* f}{\hat{\lambda}_j - \omega^2}$$

  - First $k - p$ moments of $\hat{x}$ and x match.
  - Interpolation in the p deflated eigenvalues.

Applications

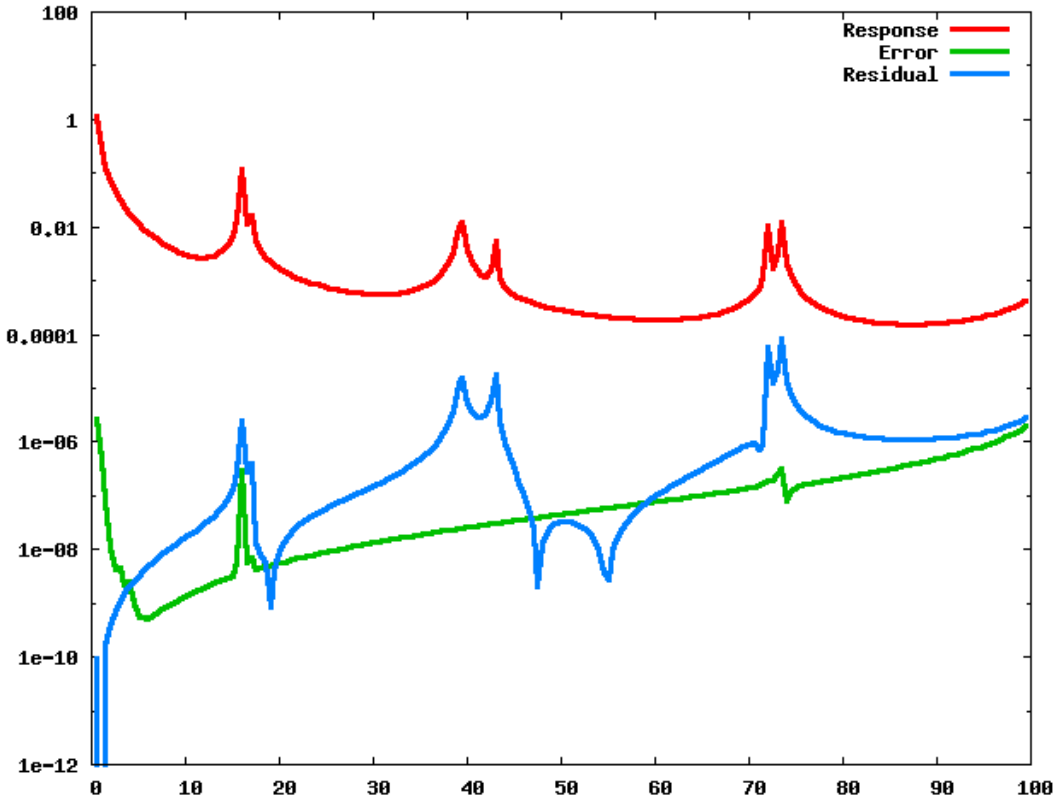
- AMLS frequency sweeping
- Multiple right-hand sides:
  - Parameterized Lanczos for right-hand side 1
  - Keep Ritz vectors
  - Recycle Ritz vectors for coming right-hand sides
- Changing σ: recycle Ritz vectors for new pole.

Windscreen

- Glaverbel-BMW windscreen
- grid : 3 layers of 60×30 HEX08 elements ($n = 22,692$)
- $\Omega = [0, 100]$
- First run:
  - unit point force at one of the corners
  - Use Lanczos method with $k = 20$ vectors.
  - We keep the Ritz values in $[0, 2 \times 100^2]$: $p = 14$

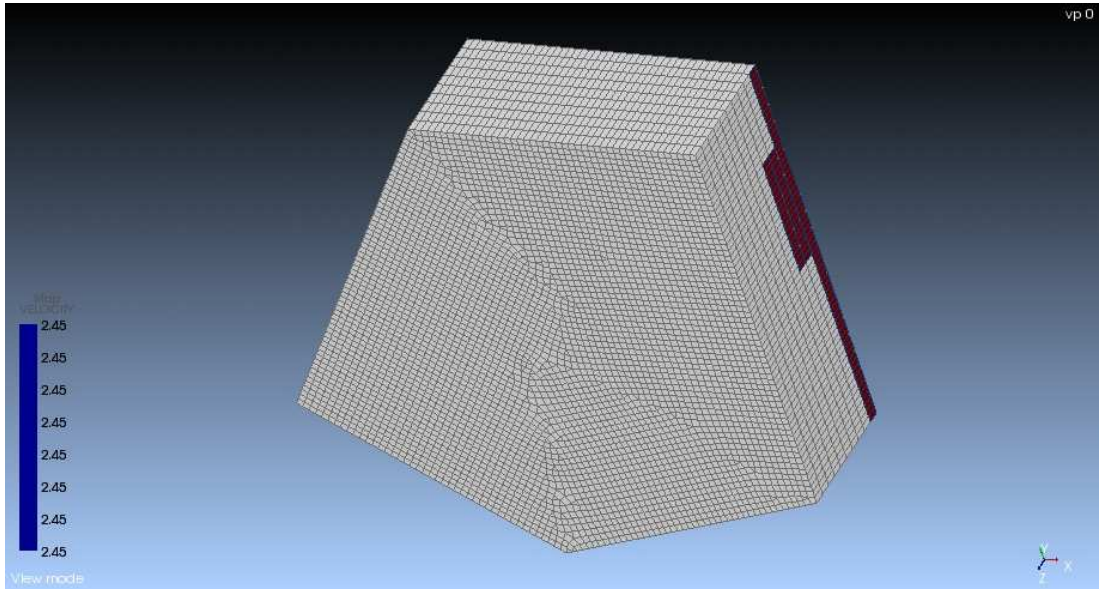
Windscreen

- Second run with other right-hand side
  - Perform 6 additional Lanczos steps



  - The largest $\kappa(\hat{B})$ is 1.9813.
  - Six iterations reduce the error in the $M\hat{B}$ norm by $2 \cdot 10^{-5}$.

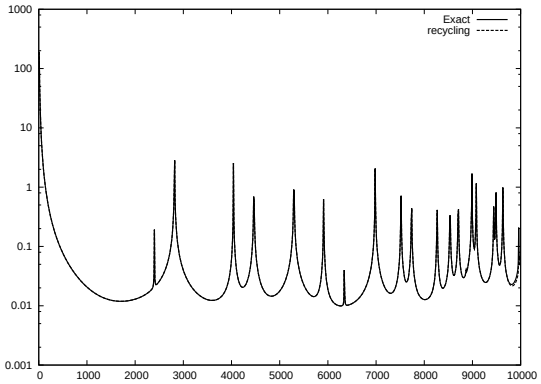
Acoustic cavity



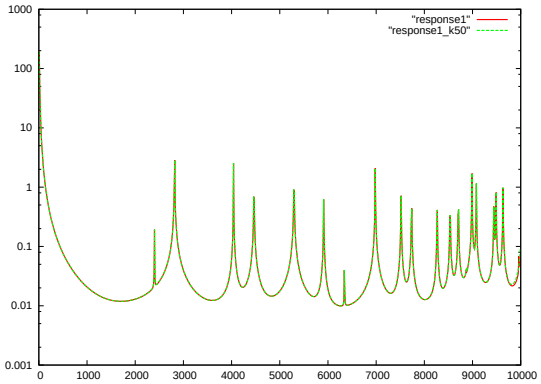
- $n = 48,158$
- Frequency range : $[0, 10000]$
- 202 right-hand sides
- matrix factorization: 8 seconds
- Lanczos method with 40 vectors: 6 seconds

Acoustic cavity (cont.)

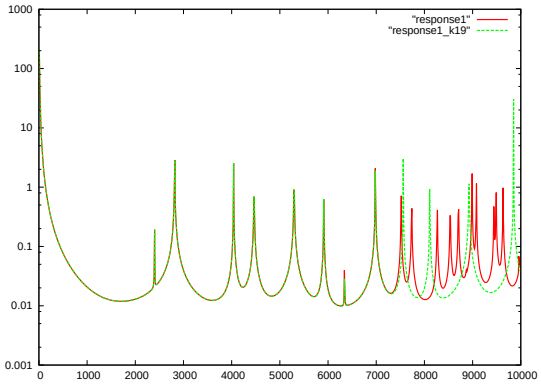
- 2nd right-hand side: keep the 31 Ritz values in $[0, 2 \times 10.000^2]$.
- 9 additional Lanczos iterations recycling 31 Ritz vectors: 2 seconds



With recycling



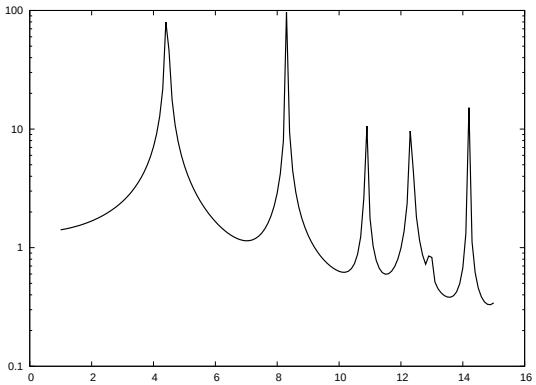
Lanczos $k = 40$



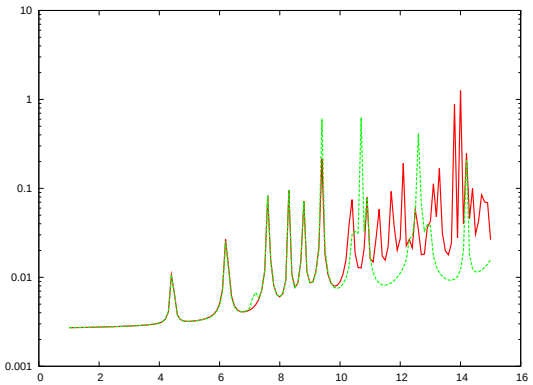
Lanczos k =9

Multiple eigenvalues

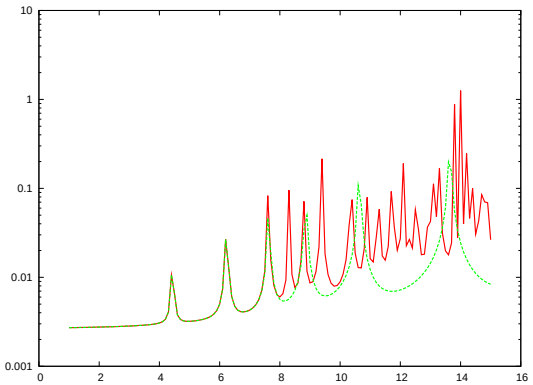
- 3D Laplacian on a cube.
- 30 Lanczos iterations with first right-hand side
- Recycle 22 Ritz pairs
- Run 8 iterations with the second right-hand side



Lanczos for f_1



Recycling for f_2



Lanczos $k = 8$ for f_2