Projected matrix equations and their application in model reduction of descriptor systems

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Matrix equations

Lyapunov equation

$$AX + XA^T = -G$$

Sylvester equation

$$AX + XF = -G$$

Bernoulli equation

$$AX + XA^T - XQX = 0$$

Riccati equation

$$AX + XA^T - XQX + G = 0$$

Lur'e equations

$$AX + XA^T + G = -KK^T$$

$$CX + H = -J^T K, \; R = JJ^T$$

Outline

- Projected Lyapunov equations
  - Balanced truncation of descriptor systems
  - Numerical solution (LR-ADI method)
- Projected Lur'e equations
  - Passivity and positive real balanced truncation
  - Contractivity and bounded real balanced truncation
- Projected Riccati equations
  - Newton's method
- Numerical examples
- Summary

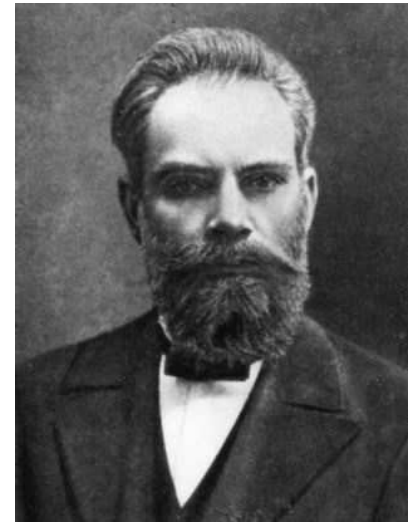
Lyapunov equations

A.M.Lyapunov, "The general problem of the stability of motion", 1892.

- $AX + XA^T = -G$ $\quad (\mathrm{Re}\,(\lambda(A)) < 0)$
  [Gantmacher'54, Daleckii/Krein'74, ...]

- $AX + XA^T = -PGP^T + (I-P)G(I-P)^T$
  [Daleckii/Krein'74, Godunov'86] $\quad (\mathrm{Re}\,(\lambda(A)) \neq 0)$

- $AXE^T + EXA^T = -G$ $\quad$ (E – nonsingular)
  [Owens/Debeljkovich'85, Chu'87, Larin'92, Penzl'98, ...]



A.M. Lyapunov
1857-1918

If E is singular, then $AXE^T + EXA^T = -G$ may have no solutions even if $\mathrm{Re}\,(\lambda(E,A)) < 0$

$$\hookrightarrow \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If solutions exist, they are always nonunique $\hookrightarrow$ $X + vv^T$, $v \in \ker E$.

Generalized Lyapunov equations

- $AXE^T + EXA^T = -EGE^T$ ($\lambda E - A$ of index 1)

  [Lewis'85, Mazko'86, Ishihara/Terra'02]

- $AX + YA^T = -G, \;\; EX = YE^T$ ($\lambda E - A$ of index 1)

  [Takaba/Morihira/Katayama'95]

- $AXE^T + EXA^T = -P_l G P_l^T, \;\; X = P_r X P_r^T$

  [St.'00,'02]

- $(aE + A)X(E - a^2 A)^T + (E - a^2 A)X(aE + A)^T = -G$

  [Müller'04] ($0 < a < 1/b^2$, $|\lambda_j(E, A)| \leq b$)

Weierstrass canonical form:

$$E = T_l \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} T_r,$$

where A_f – Jordan block ($\lambda_j(A_f)$ are finite eigenvalues of $\lambda E - A$),
N – nilpotent ($N^{\nu-1} \neq 0$, $N^\nu = 0 \;\rightsquigarrow\; \nu$ is index of $\lambda E - A$).

Projected Lyapunov equations

Consider the projected generalized Lyapunov equation

$$AXE^T + EXA^T = -P_l G P_l^T, \qquad X = P_r X P_r^T,$$

where

$$P_l = T_l \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad P_r = T_r^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_r.$$

Applications

- stability of the differential-algebraic equation $E\,\dot{x}(t) = A\,x(t)$
- inertia theory for the matrix pencil $\lambda E - A$
- control problems for $E\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$ (controllability/observability, balancing, model reduction)

Descriptor systems

Time domain representation

$$u \longrightarrow \boxed{\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}} \longrightarrow y$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$,

$u(t) \in \mathbb{R}^m$ – input, $x(t) \in \mathbb{R}^n$ – state, $y(t) \in \mathbb{R}^p$ – output.

Frequency domain representation

Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $y(t) \mapsto \boldsymbol{y}(s)$

$$\hookrightarrow \quad \boldsymbol{y}(s) = \big(C(sE-A)^{-1}B + D\big)\boldsymbol{u}(s) + C(sE-A)^{-1}Ex(0)$$

$\boldsymbol{G}(s) = C(sE-A)^{-1}B + D$ is the transfer function

Model reduction problem

Given a large-scale system

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t),$$
$$y(t) = C\,x(t) + D\,u(t)$$

with $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $n \gg m, p$,

find a reduced-order system

$$\tilde{E}\,\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t),$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t) + \tilde{D}\,u(t)$$

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{\ell,\ell}$, $\tilde{B} \in \mathbb{R}^{\ell,m}$, $\tilde{C} \in \mathbb{R}^{p,\ell}$, $\tilde{D} \in \mathbb{R}^{p,m}$, $\ell \ll n$.

- preservation of system properties
  - stability ($\lambda_j(E, A) \in \mathbb{C}^-$)
  - passivity (= system does not generate energy)
  - contractivity ($\|y\|_{\mathbb{L}_2} \leq \|u\|_{\mathbb{L}_2}$)
- small approximation error $\Longrightarrow$ need for error bounds
  $\hookrightarrow$ $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\| \leq tol$ or $\|\tilde{y} - y\| \leq tol \cdot \|u\|$ for all $u \in \mathcal{U}$
- numerically stable and efficient methods

Balanced truncation: $E = I$

- The controllability Gramian and the observability Gramian solve

$$A\mathcal{G}_c + \mathcal{G}_c A^T = -BB^T, \qquad A^T\mathcal{G}_o + \mathcal{G}_o A = -C^T C.$$

- System $\boldsymbol{G} = (A, B, C, D)$ is balanced, if $\mathcal{G}_c = \mathcal{G}_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$.

- $\sigma_j = \sqrt{\lambda_j(\mathcal{G}_c\mathcal{G}_o)}$ are the Hankel singular values

Idea: balance the system, i.e., find a state space transformation

$$(\hat{A},\ \hat{B},\ \hat{C},\ \hat{D}) = (T^{-1}AT,\ T^{-1}B,\ CT,\ D)$$
$$= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1, C_2], D \right)$$

such that $\hat{\mathcal{G}}_c = \hat{\mathcal{G}}_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and truncate the states corresponding to small σ_j $\quad\hookrightarrow\quad$ $\widetilde{\boldsymbol{G}} = (A_{11},\ B_1,\ C_1,\ D)$

Generalized Gramians

$$\boldsymbol{G}(s) = C(sE - A)^{-1}B + D = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s),$$

where $\boldsymbol{G}_{sp}(s) = C_1(sI - A_f)^{-1}B_1$ is strictly proper and

$$\boldsymbol{P}(s) = C_2(sN - I)^{-1}B_2 + D = M_0 + sM_1 + \ldots + s^{\nu-1}M_{\nu-1}.$$

- The proper Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ solve the projected generalized continuous-time Lyapunov equations

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_l BB^T P_l^T, \qquad \mathcal{G}_{pc} = P_r \mathcal{G}_{pc} P_r^T,$$
$$E^T\mathcal{G}_{po}\,A + A^T\mathcal{G}_{po}E = -P_r^T C^T C P_r, \qquad \mathcal{G}_{po} = P_l^T \mathcal{G}_{po} P_l.$$

- The improper Gramians $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ solve the projected generalized discrete-time Lyapunov equations

$$A\,\mathcal{G}_{ic}\,A^T - E\,\mathcal{G}_{ic}E^T = Q_l BB^T Q_l^T, \qquad \mathcal{G}_{ic} = Q_r \mathcal{G}_{ic} Q_r^T,$$
$$A^T\mathcal{G}_{io}\,A - E^T\mathcal{G}_{io}\,E = Q_r^T C^T C Q_r, \qquad \mathcal{G}_{io} = Q_l^T \mathcal{G}_{io} Q_l,$$

where $Q_l = I - P_l$ and $Q_r = I - P_r$.

Balanced truncation: E singular

- System $\boldsymbol{G} = (E, A, B, C, D)$ is balanced, if the Gramians satisfy

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n_f}, 0, \ldots, 0),$$

$$\mathcal{G}_{ic} = \mathcal{G}_{io} = \mathrm{diag}(0, \ldots, 0, \theta_1, \ldots, \theta_{n_\infty}).$$

- $\sigma_j = \sqrt{\lambda_j(\mathcal{G}_{pc} E^T \mathcal{G}_{po} E)}$ are proper Hankel singular values

  $\theta_j = \sqrt{\lambda_j(\mathcal{G}_{ic} A^T \mathcal{G}_{io} A)}$ are improper Hankel singular values

Idea: balance the system and truncate the states corresponding to small proper and zero improper Hankel singular values.

Example

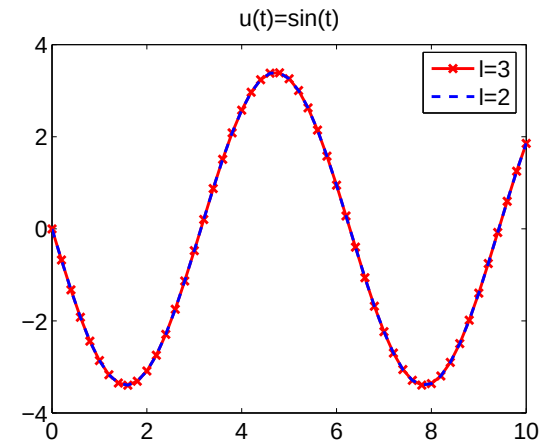
$$N\dot{x}(t) = x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0.1 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0.04 \\ 30 \\ 1 \end{bmatrix}$

Improper Hankel singular values $\theta_1 = 3.4,\ \theta_2 = 4.7 \cdot 10^{-6},\ \theta_3 = 0$

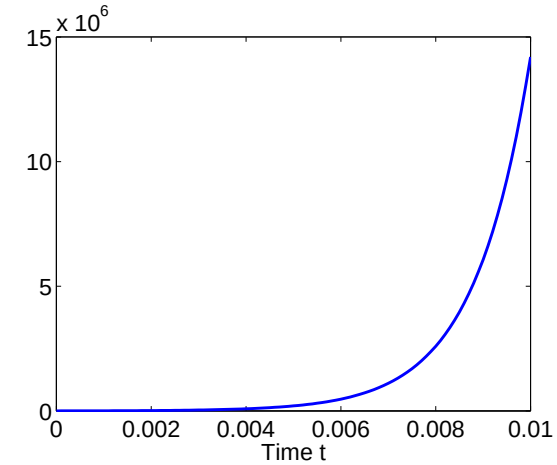
- Reduced-order system: $\ell = 2$

$$\begin{bmatrix} 1.2 & 1.2 \\ -1.2 & -1.2 \end{bmatrix} \dot{\tilde{x}}(t) = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix} \tilde{x}(t) + \tilde{B}u(t)$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t)$$



- Reduced-order system: $\ell = 1$

$$\dot{\tilde{x}}(t) = 850\,\tilde{x}(t) + 1567u(t)$$
$$\tilde{y}(t) = 1.9\,\tilde{x}(t)$$



Balanced truncation method

1. Compute

$$\mathcal{G}_{pc} = R_p R_p^T, \quad \mathcal{G}_{po} = L_p L_p^T, \quad \mathcal{G}_{ic} = R_i R_i^T, \quad \mathcal{G}_{io} = L_i L_i^T;$$

2. Compute the SVD

$$L_p^T E R_p = [U_{11},\, U_{12}] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_{11},\, V_{12}]^T;$$

3. Compute the SVD

$$L_i^T A R_i = [U_{21},\, U_{22}] \begin{bmatrix} \Theta_1 & \\ & 0 \end{bmatrix} [V_{21},\, V_{22}]^T;$$

4. $[\,\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C},\, \tilde{D}\,] = [\,W^T E T,\, W^T A T,\, W^T B,\, C T,\, D\,]$ with

$$W = [\,L_p U_{11} \Sigma_1^{-1/2},\, L_i U_{21} \Theta_1^{-1/2}], \quad T = [\,R_p V_{11} \Sigma_1^{-1/2},\, R_i V_{21} \Theta_1^{-1/2}\,].$$

Balanced truncation: properties

- $\tilde{E} = W^T E T = \begin{bmatrix} I & 0 \\ 0 & \tilde{E}_\infty \end{bmatrix}, \qquad \tilde{A} = W^T A T = \begin{bmatrix} \tilde{A}_f & 0 \\ 0 & I \end{bmatrix}$
- $\lambda \tilde{E} - \tilde{A}$ is regular and stable
- $(\,\tilde{E},\, \tilde{A},\, \tilde{B},\, \tilde{C},\, \tilde{D}\,)$ is balanced
- $\tilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s)$ is strictly proper
- Error bound:

$$\|\tilde{y} - y\|_{\mathbb{L}_2} \leq \|\tilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \|u\|_{\mathbb{L}_2} \leq 2(\sigma_{\ell_f+1} + \ldots + \sigma_{n_f}) \|u\|_{\mathbb{L}_2}$$

Solving projected Lyapunov equations

$$E\,\mathcal{G}_{pc}\,A^T + A\,\mathcal{G}_{pc}E^T = -P_l B B^T P_l^T, \qquad \mathcal{G}_{pc} = P_r \mathcal{G}_{pc} P_r^T$$

$\rightsquigarrow$ $\mathcal{G}_{pc} = R_p R_p^T,\ R_p \in \mathbb{R}^{n,n}$ or $\mathcal{G}_{pc} \approx R R^T,\ R \in \mathbb{R}^{n,r},\ r \ll n$

[✓] Hammarling method
$\hookrightarrow$ medium/dense
[Hammarling'82, Penzl'98, Kressner'06, St.'02]

[✓] Sign function method
$\hookrightarrow$ large/dense
[Roberts'71, Byers'87, Larin/Aliev'93, ..., Benner/Quintana-Ortí'99, St.'07]

[✓] ADI and Smith methods
$\hookrightarrow$ large/sparse
[Wachspress'88, Penzl'99, Li/White'02, Benner/Li/Penzl/'08, St.'08]

[✓] Krylov subspace methods
$\hookrightarrow$ large/sparse
[Saad'90, Jaimoukha/Kasenally'94] Simoncini'06, Simoncini/St.'09]

[ ] ADI + global Arnoldi,
ADI + Galerkin projection
[Jbilou'08]
[Benner/Li/Truhar'08]

Generalized ADI method

$$(E+\tau_k A)X_{k-1/2}A^T = -P_l B B^T P_l^T - AX_{k-1}(E-\tau_k A)^T$$

$$(E+\overline{\tau}_k A)\, X_k^T\, A^T = -P_l B B^T P_l^T - AX_{k-1/2}^T(E-\overline{\tau}_k A)^T$$

$$\Downarrow \quad X_k = R_k R_k^T$$

Low-rank ADI method:

$$Y_1 = \sqrt{-2\mathrm{Re}\,(\tau_1)}\,(E+\tau_1 A)^{-1}P_l B, \qquad R_1 = Y_1,$$

$$Y_k = \sqrt{\frac{\mathrm{Re}\,(\tau_k)}{\mathrm{Re}\,(\tau_{k-1})}}\,\big(Y_{k-1} - (\overline{\tau}_{k-1}+\tau_k)(E+\tau_k A)^{-1}AY_{k-1}\big),$$

$$R_k = [\,R_{k-1}, \quad Y_k\,]$$

- sequence of low-rank factors $R_k \in \mathbb{R}^{n,km}$ such that $R_k = P_r R_k$
- (sub)optimal shift parameters [Penzl'98, Sabino'06, Benner et al.'08]
- solve $(E+\tau_k A)x = P_l b$

Computing the projectors

For computing the projectors P_l and P_r use the structure of E and A

[✓] semi-explicit systems (index 1)

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

[✓] Stokes-like systems (index 2)

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

[✓] mechanical systems (index 3)

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I & 0 \\ D & K & -G^T \\ G & 0 & 0 \end{bmatrix}$$

[✓] electrical circuits (index 1 and 2) [St.'08, Reis/St.'08]

Remark: For some problems the explicit computation of the projectors can be avoided [Heinkenschloss/Sorensen/Sun'08, Freitas/Rommes/Martins'08]

Lur'e equations

- (also Kalman-Yacubovich-Popov equations)

  [Ionescu/Oara/Weiss'99, Phillips/Daniel/Silveira'03]

$$AX + XA^T + G = -KK^T,$$
$$CX + H = -J^T K, \quad R = JJ^T$$

A.I. Lur'e
1901-1980

- Projected Lur'e equations [Reis/St.'08]

$$AXE^T + EXA^T + P_l G P_l^T = -KK^T, \quad X = P_r X P_r^T$$
$$CXE^T + HP_l^T = -JK^T, \quad R = JJ^T$$

Applications

- passivity and positive real balanced truncation
- contractivity and bounded real balanced truncation

Passivity via Lur'e equations

$(E,\, A,\, B,\, C,\, D\,)$ is passive (= does not generate energy)

$\Longleftrightarrow$ $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is positive real, i.e.,

$\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$ and $\boldsymbol{G}(s) + \boldsymbol{G}^T(\overline{s}) \geq 0$ for all $s \in \mathbb{C}^+$

$\Longleftrightarrow$ $\boldsymbol{G}(s) = \boldsymbol{G}_{sp}(s) + M_0 + M_1 s$, where $M_1 = M_1^T \geq 0$ and the projected Lur'e equations

$$\begin{aligned} AXE^T + EXA^T &= -K_c K_c^T \\ CXE^T - B^T P_l^T &= -J_c K_c^T \\ X = P_r X P_r^T,\ M_0 + M_0^T &= J_c J_c^T \end{aligned}$$

$$\begin{aligned} A^T Y E + E^T Y A &= -K_o^T K_o \\ B^T Y E - C P_r &= -J_o^T K_o \\ Y = P_l^T Y P_l,\ M_0 + M_0^T &= J_o^T J_o \end{aligned}$$

are solvable for $X = X^T \geq 0$, J_c, K_c and $Y = Y^T \geq 0$, J_o, K_o.

$\hookrightarrow$ $0 \leq X_{\min} \leq X \leq X_{\max}$, $X_{\min}$ is positive real controllability Gramian

$0 \leq Y_{\min} \leq Y \leq Y_{\max}$, $Y_{\min}$ is positive real observability Gramian

Conractivity via Lur'e equations

$(E,\, A,\, B,\, C,\, D\,)$ is contractive $(\|y\|_{\mathbb{L}_2} \le \|u\|_{\mathbb{L}_2})$

$\iff$ $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is bounded real, i.e.,

$\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$ and $I - \boldsymbol{G}(s)\,\boldsymbol{G}^T(\overline{s}) \ge 0$ for all $s \in \mathbb{C}^+$

$\iff$ $\boldsymbol{G}(s) = \boldsymbol{G}_{sp}(s) + M_0$ and the projected Lur'e equations

$$\begin{aligned} AXE^T + EXA^T + P_l BB^T P_l^T &= -K_c K_c^T \\ CXE^T + M_0 B^T P_l^T &= -J_c K_c^T \\ X = P_r X P_r^T,\ \ I - M_0 M_0^T &= J_c J_c^T \end{aligned}$$

$$\begin{aligned} A^T YE + E^T YA + P_r^T C^T C P_r &= -K_o^T K_o \\ B^T YE + M_0^T C P_r &= -J_o^T K_o \\ Y = P_l^T Y P_l,\ \ I - M_0^T M_0 &= J_o^T J_o \end{aligned}$$

are solvable for $X = X^T \ge 0$, J_c, K_c and $Y = Y^T \ge 0$, J_o, K_o.

$\hookrightarrow$ $0 \le X_{\min} \le X \le X_{\max}$, $X_{\min}$ is bounded real controllability Gramian

$0 \le Y_{\min} \le Y \le Y_{\max}$, $Y_{\min}$ is bounded real observability Gramian

PR / BR balanced truncation method

- Compute $X_{\min} = R_1 R_1^T$, $Y_{\min} = L_1 L_1^T$ (= solve Lur'e equations);
- Compute $\mathcal{G}_{ic} = R_2 R_2^T$, $\mathcal{G}_{io} = L_2 L_2^T$ (= solve Lyapunov equations);
- Compute the SVD $L_1^T E R_1 = [U_{11},\, U_{12}] \begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix} [V_{11},\, V_{12}]^T$;
- Compute the SVD $L_2^T A R_2 = [U_{21},\, U_{22}] \begin{bmatrix} \Theta_1 & \\ & 0 \end{bmatrix} [V_{21},\, V_{22}]^T$;
- Compute $\widetilde{\boldsymbol{G}} = [\, W^T E T,\ W^T A T,\ W^T B,\ C T,\ D \,]$ with

  $W = [\, L_1 U_{11} \Pi_1^{-1/2},\ L_2 U_{21} \Theta_1^{-1/2} \,]$, $T = [\, R_1 V_{11} \Pi_1^{-1/2},\ R_2 V_{21} \Theta_1^{-1/2} \,]$.

Properties

- Positive real balanced truncation
  - $\widetilde{\boldsymbol{G}} = (\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D})$ is passive
  - Error bound:

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_\infty \leq 2\,\|(M_0 + M_0^T)^{-1}\|\|\boldsymbol{G} + M_0^T\|_\infty \|\widetilde{\boldsymbol{G}} + M_0^T\|_\infty \sum_{j=\ell_f+1}^{n_f} \pi_j$$

- Bounded real balanced truncation
  - $\widetilde{\boldsymbol{G}} = (\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D})$ is contractive
  - Error bound: $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_\infty \leq 2(\pi_{\ell_f+1} + \ldots + \pi_{n_f})$

Solving projected Lur'e equations

$$AXE^T + EXA^T + P_l BB^T P_l^T = -KK^T, \quad X = P_r X P_r^T,$$
$$CXE^T + HP_l^T = -JK^T, \quad R = JJ^T$$

- R – nonsingular $\Longrightarrow$ projected Riccati equation

$$AXE^T + EXA^T + P_l BB^T P_l^T + (CXE^T + HP_l^T)^T R^{-1} (CXE^T + HP_l^T) = 0,$$
$$X = P_r X P_r^T$$

  [✓] Newton's method [Benner/St.'08]

  [✓] Generalized Hamiltonian eigenvalue problem [Reis/St.'08]

  [ ] Krylov subspace methods [Jaimoukha/Kasenally'94, Benner'97, ..., Jbilou'06,09]

- R – singular
  - small/dense problems: reduce to the Riccati equation of smaller dimension [Weiss/Wang/Speyer'94]
  - large/sparse problems – ?

Newton's method

$$\mathcal{R}(X) = AXE^T + EXA^T + EXC^TCXE^T + P_lBB^TP_l^T = 0, \quad X = P_rXP_r^T$$

- Let $\mathbb{S}_P = \{ X \in \mathbb{R}^{n,n} \ : \ X = X^T, \ X = PXP^T \}$ for a projector P.
  Frechét derivative of $\mathcal{R} : \mathbb{S}_{P_r} \to \mathbb{S}_{P_l}$ at X is given by

  $$\mathcal{R}'_X(Z) = (A + EXC^TCP_r)\, Z\, E + E\, Z\, (A + EXC^TCP_r)^T$$

- Newton's method: $X_{k+1} = X_k - (\mathcal{R}'_{X_k})^{-1}(\mathcal{R}(X_k))$.

  FOR $k = 0, 1, 2, \ldots$

  - Compute $K_k = EX_kC^T$ and $A_k = A + K_kCP_r$
  - Solve the projected Lyapunov equation

    $$A_kZ_kE^T + EZ_kA_k^T = -P_l\mathcal{R}(X_k)P_l^T, \quad Z_k = P_rZ_kP_r^T.$$

  - Compute $X_{k+1} = X_k + Z_k$.

  END FOR

Newton's method: properties

- $\lambda E - A_k$ are stable and $X_k = P_r X_k P_r^T$
- $\lim_{k\to\infty} \mathcal{R}(X_k) = 0$
- $\lim_{k\to\infty} X_k = X_{\min}$
  (quadratically if $\lambda E - (A - E X_{\min} C^T C P_r)$ is stable)
- solve projected Lyapunov equations via the generalized ADI method

  $\hookrightarrow$ $(E + \tau A_k)^{-1} = \big((E + \tau A) - (\tau K_k)(C P_r)\big)^{-1}$ is required

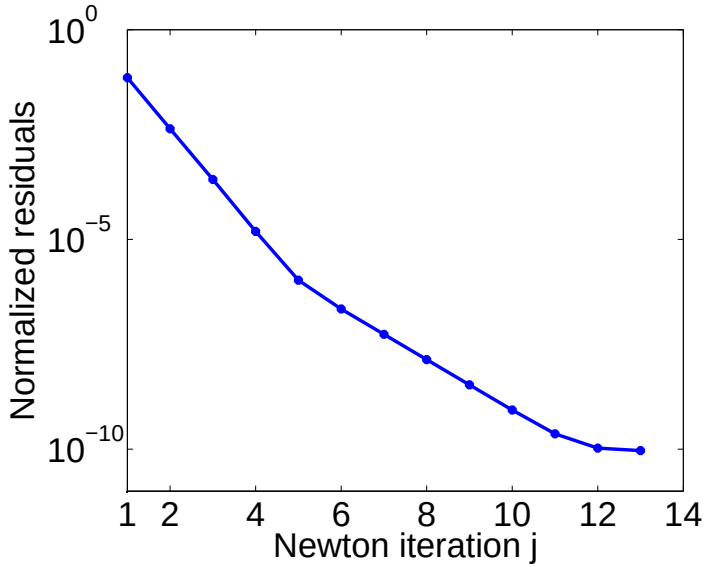
  $\hookrightarrow$ use "sparse + low-rank" structure of $E + \tau A_k$ and the Sherman-Morrison-Woodbury formula

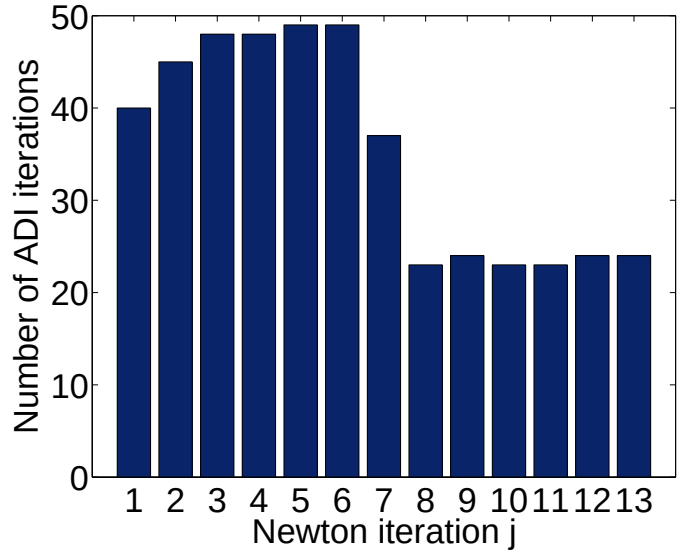
  $$(E + \tau A_k)^{-1} = \hat{A}^{-1} + \tau \hat{A}^{-1} K_k (I_p - \tau C P_r \hat{A}^{-1} K_k)^{-1} C P_r \hat{A}^{-1}$$

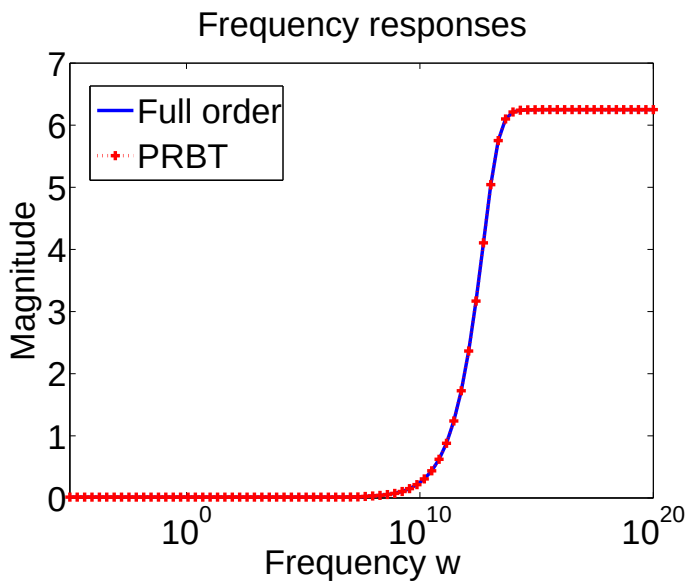
  with sparse $\hat{A} = E + \tau A$
- computing the approximate factored solution $X_{\min} \approx R_k R_k^T$ is possible using the generalized LR-ADI method

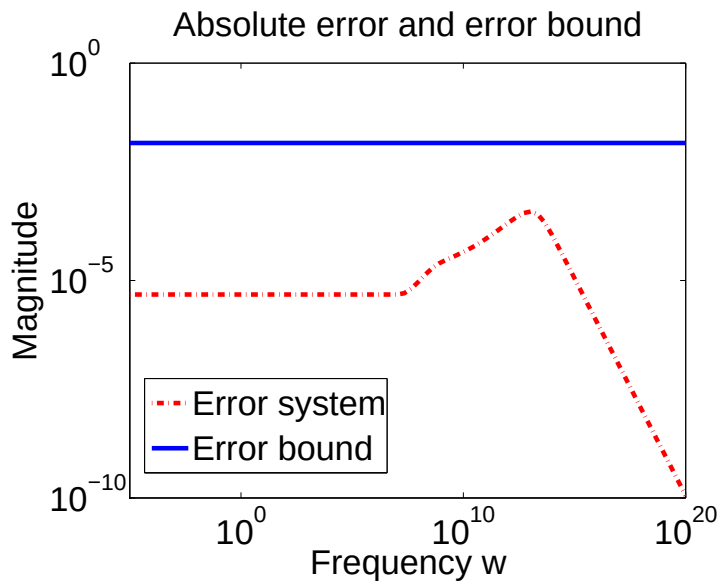
RC circuit: index 1 – PRBT

- $n = 2007, \ m = 3, \ p = 3 \quad \Longrightarrow \quad \ell = 42$



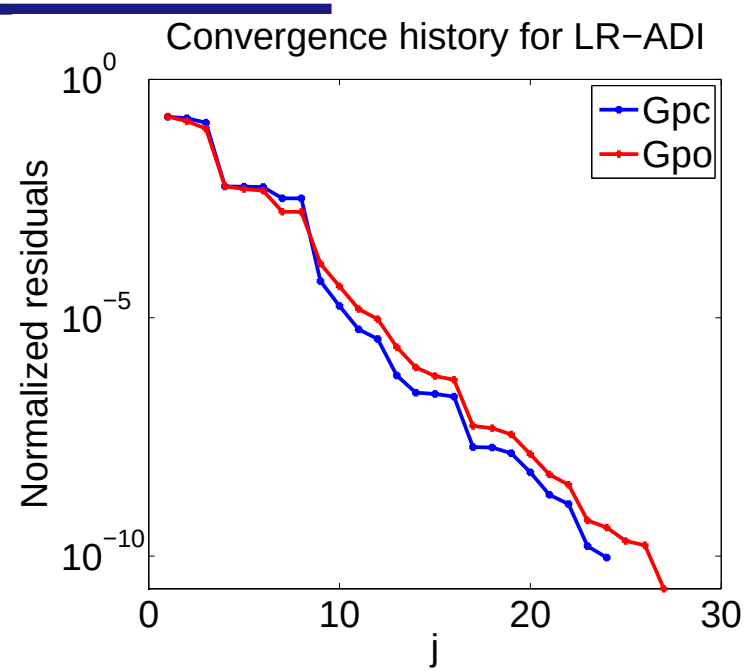


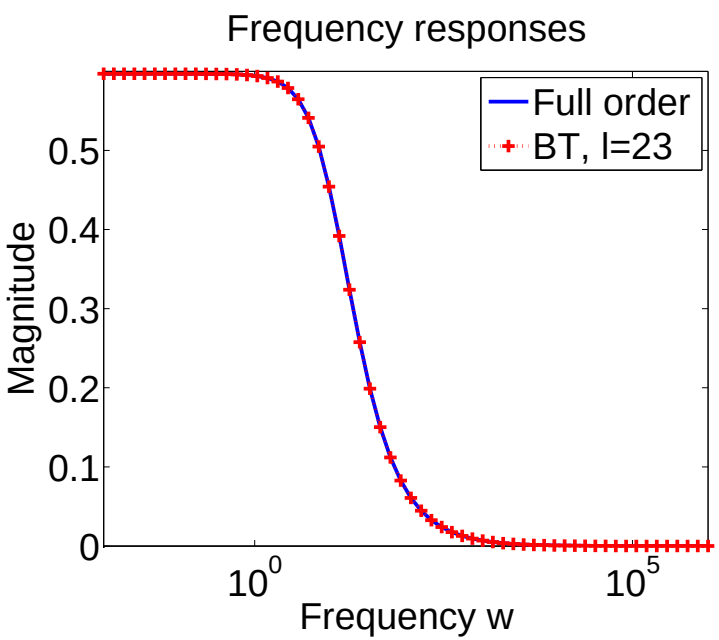


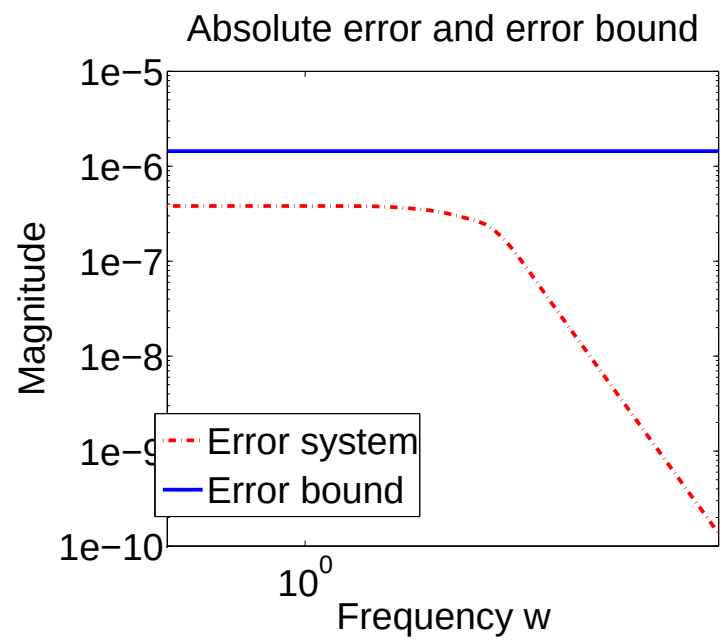


Stokes equation: index 2 - BT

- $n = 29799, \;\; m = 5, \;\; p = 5$
- $\mathcal{G}_{pc} \approx \tilde{R}_p \tilde{R}_p^T, \quad \tilde{R}_p \in \mathbb{R}^{n,120}$
- $\mathcal{G}_{po} \approx \tilde{L}_p \tilde{L}_p^T, \quad \tilde{L}_p \in \mathbb{R}^{n,135}$
- Reduced system: $\ell = 23$



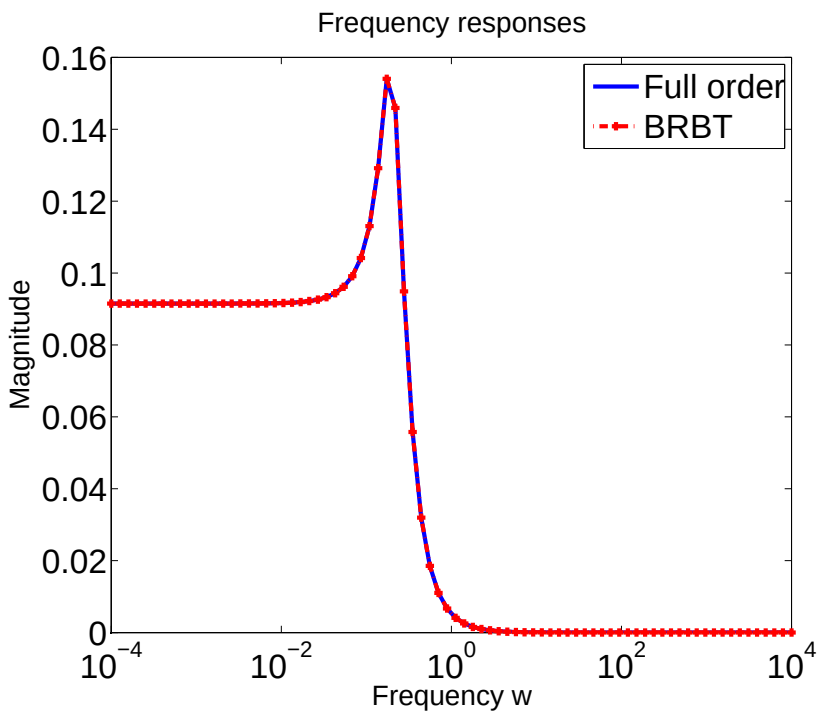


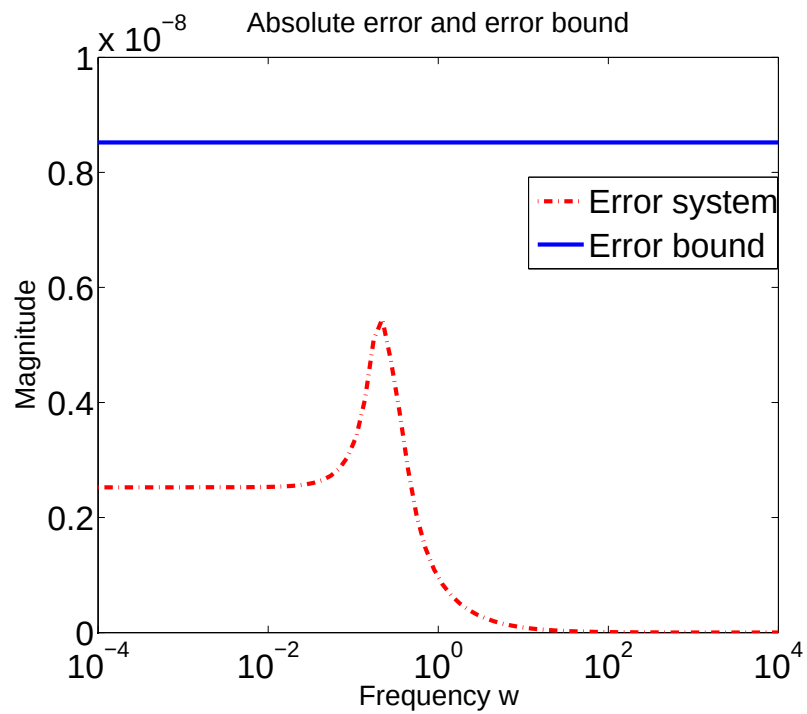


Mechanical system: index 3 – BRBT

- $n = 60001, \quad m = p = 1 \quad \Longrightarrow \quad \ell = 20$

	# Newton	$\|\mathcal{R}(X)\|/\|PGP^T\|_F$	# LR-ADI	rank	CPU (sec.)
$X_{\min}$	3	$6.71 \cdot 10^{-9}$	16	19	93.14
$Y_{\min}$	2	$8.74 \cdot 10^{-8}$	18	17	63.52





Summary and open problems

- Projected matrix equations (Lyapunov, Lur'e, Riccati) are useful tools in control problems for descriptor systems
  - stability
  - passivity (Positive Real Lemma)
  - contractivity (Bounded Real Lemma)
  - balancing-related model reduction
- Projectors P_l and P_r are required $\Rightarrow$ use the structure of E, A
- Implementation of the solvers for large-scale projected matrix equations will be included (hopefully soon) in MATLAB Toolbox MESS-Matrix Equations Sparse Solvers [Saak/Mena/Benner]

Open problems

- Numerical solution of large-scale projected Lur'e equations
- Computation of stabilizing initial guess in Newton's method for large-scale projected Riccati equations