On the inverse eigenvalue problem for T-alternating and T-palindromic matrix polynomials

Leonhard Batzke and Christian Mehl

In the proof of [1, Theorem 1.7], there is a hidden matrix polynomial triangularization result: If $\mathbb{F}$ is an algebraically closed field, then any $P(\lambda) \in \mathbb{F}[\lambda]^{n,n}$ with nonsingular leading coefficient is equivalent to a monic upper triangular matrix polynomial of the same degree. Recently, this result has been generalized twice, in [5] to regular quadratic matrix polynomials and then in [4] to arbitrary $n \times m$ matrix polynomials with $n \leq m$.

Also, structured matrix polynomials have been focused on recently. A matrix polynomial $P(\lambda)$ is called T-even if $P(\lambda) = P(-\lambda)^T$ and T-odd if $-P(\lambda) = P(-\lambda)^T$, generalizing the notion of even and odd scalar polynomials. In [2, Theorem 3.10 and 3.11], necessary and sufficient conditions for a matrix polynomial to be the Smith form of *some* T-alternating (i.e., T-even or T-odd) matrix polynomial, are presented, resulting in certain pairing conditions on the elementary divisors. However, given the Smith form $S(\lambda)$ of some T-alternating matrix polynomial, the T-alternating matrix polynomial with Smith form $S(\lambda)$ constructed in [2] is not in triangular form and in general has high degree resulting in many infinite elementary divisors. Similar results were obtained in [3] for T-palindromic matrix polynomials.

Note that the trangularization results cited in the first paragraph each solve an inverse polynomial eigenvalue problem: Given a certain list of elementary divisors (finite and infinite), is there a matrix polynomial that has these elementary divisors? Since for unstructured matrix polynomials, this problem was answered affirmatively in [4], we want to tackle the inverse eigenvalue problem for T-alternating matrix polynomials left open in [2]: Given a list of elementary divisors satisfying the pairing conditions to possibly be realized in a T-alternating matrix polynomial, is there a T-alternating matrix polynomial in triangular form with these elementary divisors and of minimal degree?

It is clear that we cannot transform all coefficient matrices into triangular form using strict equivalence transformations, instead unimodular equivalences are used, i.e., $P(\lambda)$ is unimodularly equivalent to $Q(\lambda)$ if and only if $P(\lambda) = E(\lambda)Q(\lambda)F(\lambda)$ for some $E(\lambda), F(\lambda)$ that have constant nonzero determinants. Unfortunately, examples show that infinite elementary divisors are not preserved under unimodular equivalences. There is the notion of strong unimodular equivalence, i.e., $P(\lambda)$ and $Q(\lambda)$ are unimodularly equivalent *and* they have the same eigenstructure at ∞, but a canonical form under this relation has not yet been developed.

Hence, in our results we restrict ourselves to matrix polynomials with nonsingular leading coefficient and ones unimodularly equivalent to them, thus satisfying $\deg\big(\det P(\lambda)\big) = nk$ for some positive integer k. Observe that under this condition, the lowest possible degree of a matrix polynomial with the elementary divisors of $P(\lambda)$ is k. The main result is then given by:

Let the field $\mathbb{F}$ be algebraically closed and let $S(\lambda) \in \mathbb{F}[\lambda]^{n,n}$ be the Smith form of some T-even or T-odd matrix polynomial. If $\deg\big(\det S(\lambda)\big) = nk$, then $S(\lambda)$ is unimodularly equivalent to a lower anti-triangular $n \times n$ matrix polynomial of degree k that is T-even or T-odd, respectively.

Note that we aim for an anti-triangular form of the coefficient matrices, i.e., for a matrix $A = [a_{ij}]$ it is $a_{ij} = 0$ for all $i + j \leq n$, since an upper triangular form of the coefficient matrices of a T-alternating matrix polynomial that are symmetric / skew-symmetric would mean that they are diagonal, which is too restrictive.

Finally, we are able to generalize the obtained results to T-palindromic (i.e., $P(\lambda) = \operatorname{rev} P(\lambda)^T$, where $\operatorname{rev} P(\lambda)$ is obtained from $P(\lambda)$ by reversing the order of the coefficient matrices) and T-anti-palindromic (i.e., $P(\lambda) = -\operatorname{rev} P(\lambda)^T$) matrix polynomials using Cayley transformations of matrix polynomials.

References

[1] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[2] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.

[3] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms for palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[4] L. Taslaman, F. Tisseur, and I. Zaballa. Triangularizing matrix polynomials. *Linear Algebra Appl.*, 439:1679–1699, 2013.

[5] F. Tisseur and I. Zaballa. Triangularizing quadratic matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 34:312–337, 2013.

Contour methods for holomorphic operator pencils with an application to eigenvalue problems on unbounded domains

Wolf-Jürgen Beyn, Yury Latushkin and Jens Rottmann-Matthes

Our contribution has two parts. In the first we discuss abstract nonlinear eigenvalue problems

$$F(\lambda)v = 0, \quad \lambda \in \Omega \subset \mathbb{C}, \tag{1}$$

where $F(\lambda), \lambda \in \Omega$ is a holomorphic family of Fredholm operators of index 0. We extend the notion of an invariant pair from [3],[5],[6] to this setting and state a suitable generalization of Keldysh' theorem. In the second part, based on [2],[4], we generalize a contour method for locating isolated eigenvalues and associated invariant pairs for the system (1). Then the method is applied to nonlinear eigenvalue problems which arise in the stability analysis of waves in parabolic systems.

It is interesting and numerically relevant to generalize the notion of an invariant subspace from the linear to the nonlinear case. Let $F(\lambda) : \mathcal{H} \mapsto \mathcal{K}$ be Fredholm operators from a Banach space $\mathcal{H}$ into a Banach space $\mathcal{K}$, which depend holomorphically on λ in some domain $\Omega \subset \mathbb{C}$ and for which $F(\lambda)$ is invertible for at least one $\lambda \in \Omega$. We define a pair (X, Λ) of linear operators $X \in L[\mathbb{C}^k, \mathcal{H}]$ and $\Lambda \in \mathbb{C}^{k\times k}$ to be *right invariant* if

$$\int_\Gamma F(z)X(zI_k - \Lambda)^{-1}dz = 0$$

for some simple closed contour Γ in Ω which encloses the spectrum $\sigma(\Lambda)$. As in [3],[5],[6] we call the pair *minimal*, if the map $X \otimes (\lambda I_k - \Lambda) \in L[\mathbb{C}^k, \mathcal{H} \times \mathbb{C}^k]$ is one to one for all $\lambda \in \Omega$, and a minimal pair is called *simple* if it cannot be extended to minimal pair of order $k+1$ with the same spectrum $\sigma(\Lambda)$. Similar notions hold for *left invariant* pairs $(Y, \Lambda) \in L[\mathcal{K}, \mathbb{C}^k] \times \mathbb{C}^{k\times k}$ defined by $\int_\Gamma (zI_k - \Lambda)^{-1}YF(z)dz = 0$.

A suitable generalization of Keldysh' Theorem states that for every simple right invariant pair (X, Λ), there exists a neighborhood $\mathcal{U}$ of $\sigma(\Lambda)$ in Ω, a simple left invariant pair (Y, Λ) and a holomorphic function $H : \mathcal{U} \mapsto L[\mathcal{K}, \mathcal{H}]$ such that

$$F(\lambda)^{-1} = X(\lambda I_k - \Lambda)^{-1}Y + H(\lambda), \quad \lambda \in \mathcal{U} \setminus \sigma(\Lambda).$$

The formula suggests the following generalization of the contour method in [1],[2],[4] for computing all eigenvalues of (1) inside the contour Γ as well as approximate eigenvectors. Take a set of m test functionals $W \in L[\mathcal{H}, \mathbb{C}^m]$ and ℓ right hand sides $R \in L[\mathbb{C}^\ell, \mathcal{K}]$ and compute the contour integrals

$$D_j = \frac{1}{2\pi i}\int_\Gamma \lambda^j WF(\lambda)^{-1}Rd\lambda \in \mathbb{C}^{m\times\ell}, \quad j = 0, 1. \tag{2}$$

Assuming $k \le \ell \le m$ and the matrices WX, YR to have maximum rank, one can determine the number k of eigenvalues inside Γ from a rank revealing SVD of D_0. Furthermore, combining this with the matrix D_1 allows to compute an approximation of the invariant pair (X, Λ), cf. [2],[4].

The method is particularly well suited for localizing a few isolated eigenvalues within a given contour when there is a large (e.g. essential) spectrum outside the contour. We apply the method to the stability problem for traveling waves $u(x, t) = v(x - ct)$ of a parabolic system

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R},\ t \ge 0,\ u(x, t) \in \mathbb{R}^N,\ f : \mathbb{R}^N \mapsto \mathbb{R}^N \text{ smooth.}$$

The linear eigenvalue problem for the all line differential operator $\mathcal{L}u = Au_{xx} + cu_x + Df(v)u$ is approximated on a finite but large interval $[x_-, x_+]$ by using so called projection boundary conditions. These boundary conditions allow to reduce the length $x_+ - x_-$ of the interval considerably, but lead to a nonlinear eigenvalue problem. We provide some numerical results for the FitzHugh-Nagumo system and discuss the errors caused by this truncation and by the quadrature method involved in evaluating (2). The overall method is an alternative to the numerical computation of zeroes of the Evans function, which is a popular tool for analyzing the stability of traveling waves.

References

[1] J. Asakura, T. Sakurai, H. Tadano, T. Ikegami and K. Kimura. A numerical method for nonlinear eigenvalue problems using contour integrals. *JSIAM Letters* 1:52–55, 2009.

[2] W.-J. Beyn. An integral method for solving nonlinear eigenvalue problems. *Linear Algebra Appl.*, 436:3839–3863, 2012.

[3] W.-J. Beyn, C. Effenberger and D. Kressner. Continuation of eigenvalues and invariant pairs for parameterized nonlinear eigenvalue problems. *Numer. Math.*, 119:489–516, 2011.

[4] W.-J. Beyn, Y. Latushkin and J. Rottmann-Matthes. Finding eigenvalues of holomorphic Fredholm operator pencils using boundary value problems and contour integrals. *Integr. Equ. Oper. Theory*, 78:155-211, 2014.

[5] C. Effenberger. Robust solution methods for nonlinear eigenvalue problems. *PhD Thesis*, École polytechnique fédéral de Lausanne, 2013.

[6] D. Kressner. A block Newton method for nonlinear eigenvalue problems. *Numer. Math.*, 114:355-372, 2009.

Eigenvalue Localization and Applications

David Bindel, Amanda Hood

Simple models of vibrating systems typically involve idealized boundary conditions: elastic bodies resting on perfect anchors, electromagnetic waves in perfectly conducting boxes, etc. However, in many applications one needs information about how waves can exit the domain of interest, to be absorbed or to radiate away into space. The absorbing boundary models that account for this effect often depend nonlinearly on frequency; hence, these models often lead to nonlinear eigenvalue problems, i.e. finding λ such that $T(\lambda)$ is singular, where $T : \Omega \to \mathbb{C}^{n\times n}$ is an analytic function on a simply connected domain Ω that we assume to be regular. In our talk, we describe some recent results on the localization of eigenvalues of the nonlinear eigenvalue problem [1]. These results generalize classic theorems for the localization of linear eigenvalues, such as the Gershgorin circle theorem and Bauer-Fike theorem. The key technique in these results is generally useful: construct an artificial homotopy between two eigenvalue problems, then relate the eigenvalues of the two problems through continuity of the eigenvalues.

As with the standard eigenvalue problem, these theorems can be used to deduce crude information about eigenvalues useful for diagnosing stability of linear systems or selecting good parameters for eigenvalue computation routines. They can be used as a tool to analyze the approximation error in different numerical methods. In addition, by using a spectral Schur complement, we can relate models posed on entirely different domains. In particular, we describe how to relate eigenvalue problems from models of the same device attached to a perfect anchor (a Dirichlet boundary condition), to an infinite compliant substrate (described by a DtN map), and to a more realistic finite-size substrate model. We show some preliminary results that illustrate how such comparisons can help us quantify the approximation error associated with simplified approximate substrate models,.

References

[1] D. Bindel and A. Hood, Localization theorems for nonlinear eigenvalue problems, *SIAM J. Matrix Anal. Appl.*, (in press).

Some results on invariant pairs for matrix polynomials

Moulay Barkatou, Paola Boito and Esteban Segura Ugalde

In the present work, we focus on the notion of invariant pairs for matrix polynomials ([2], [3]). Given a matrix polynomial

$$P(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_\ell\lambda^\ell, \tag{1}$$

an *invariant pair* for $P(\lambda)$ is $(X, S) \in \mathbb{C}^{n\times k} \times \mathbb{C}^{k\times k}$ such that

$$\mathbb{P}(X, S) := A_\ell X S^\ell + \cdots + A_2 X S^2 + A_1 X S + A_0 X = 0, \tag{2}$$

where we assume $A_i \in \mathbb{C}^{n\times n}$, $i = 0, \ldots, \ell$, and k is an integer between 1 and n.

Invariant pairs are a generalization of the notion of eigenpair for matrix polynomials and are closely related to the notion of standard pairs (see [5]). From a numerical point of view, invariant pairs provide a more stable approach to the computation of several eigenvalues and eigenvectors of a given polynomial, particularly when the eigenvalues to be computed are clustered. Invariant pairs also have an interest from a purely symbolic point of view, in the context of differential systems.

1 A contour-integral approach

An equivalent representation for (2) is:

$$\mathbb{P}(X, S) := \frac{1}{2\pi i} \oint_\Gamma P(\lambda)X(\lambda I - S)^{-1} d\lambda = 0, \tag{3}$$

where $\Gamma \subseteq \mathbb{C}$ is a contour with the spectrum of S in its interior.

This formulation allows us to choose the contour Γ to compute eigenvalues lying in a particular region of the complex plane. Moreover, it makes it easier to compute a condition number and a backward error, which play an important role for a symbolic-numeric approach. For these reasons, in this part of our work we study invariant pairs from the point of view of the contour integral.

Condition number. We measure relative perturbations normwise by

$$\epsilon = \| \begin{bmatrix} \alpha_\ell^{-1}\Delta A_\ell & \alpha_{\ell-1}^{-1}\Delta A_{\ell-1} & \ldots & \alpha_1^{-1}\Delta A_1 & \alpha_0^{-1}\Delta A_0 \end{bmatrix} \|_F$$

where the α_i's are nonnegative parameters for $i = 0, \ldots, \ell$. Computations yield

$$\frac{\|[\|\Delta X\|_F, \|\Delta S\|_F]\|_2}{\|[\|X\|_F, \|S\|_F]\|_2} \leq \Psi(X, S)\epsilon + O(\epsilon^2)$$

where

$$\Psi(X, S) = \frac{\left\| \begin{bmatrix} N & M \end{bmatrix}^+ \begin{bmatrix} \alpha_\ell B_\ell & \alpha_{\ell-1}B_{\ell-1} & \ldots & \alpha_1 B_1 & \alpha_0 B_0 \end{bmatrix} \right\|_2}{\|[\|X\|_F, \|S\|_F]\|_2} \tag{4}$$

is the condition number, and

$$N = \frac{1}{2\pi\imath}\oint_\Gamma \left((\lambda I - S)^{-1}\right)^T \otimes P(\lambda)d\lambda,\ N \in \mathbb{C}^{kn\times kn}$$
$$M = \frac{1}{2\pi\imath}\oint_\Gamma \left((\lambda I - S)^{-1}\right)^T \otimes \left(P(\lambda)X(\lambda I - S)^{-1}\right)d\lambda,\ M \in \mathbb{C}^{kn\times k^2}$$
$$B_j = \frac{1}{2\pi\imath}\oint_\Gamma \left(\lambda^j X(\lambda I - S)^{-1}\right)^T \otimes I_n d\lambda, \quad j = 0,\dots,\ell, \quad B_j \in \mathbb{C}^{kn\times n^2}$$

Backward error. We obtain the following bounds for the backward error:

$$\eta(Y,T) \geq \frac{\|\frac{1}{2\pi\imath}\oint_\Gamma P(\lambda)Y(\lambda I - T)^{-1}d\lambda\|_F}{(\alpha_\ell^2\|YT^\ell\|_F^2 + \cdots + \alpha_0^2\|Y\|_F^2)^{1/2}},$$

$$\eta(Y,T) \leq \frac{\|\frac{1}{2\pi\imath}\oint_\Gamma P(\lambda)Y(\lambda I - T)^{-1}d\lambda\|_F}{(\alpha_\ell^2\sigma_{\min}(YT^\ell)^2 + \cdots + \alpha_0^2\sigma_{\min}(Y)^2)^{1/2}}$$

where α_i are nonnegative parameters.

2 Numerical method and results

In order to compute invariant pairs numerically, we experimented with Ntewton's method with line search and including Šamanskii's technique. So, $(\Delta X, \Delta S)$ is given by

$$\mathbb{P}(X,S) + t\mathbb{DP}_{(X,S)}(\Delta X, \Delta S) = 0,$$

where t solves the minimization problem with cost function

$$p(t) = \|\mathbb{P}(X + t\Delta X, S + \Delta S)\|_F^2,$$

and $\mathbb{DP}_{(X,S)}(\Delta X, \Delta S)$ is the derivative of $\mathbb{P}$.

Algorithm.
Input: initial approximation (X_0, S_0), tolerances $\varepsilon, \varepsilon_0$, a contour Γ. **Output:** an (approximated) invariant pair (X, S) with the eigenvalues of S in Γ.

1. Set $k = 0$.
2. If $\frac{\|\mathbb{P}(X,S)\|_F}{\|X\|_F} < \varepsilon$, STOP.
3. Solve $\mathbb{DP}_{(X,S)}(\Delta X_k, \Delta S_k) = -(\mathbb{P}(X_k, S_k), 0)$.
4. If $\frac{\|\mathbb{P}(X,S)\|_F}{\|X\|_F} < \varepsilon_0$ go to 7.
5. Find t by solving $\min_{t\in[0,2]} \|\|\mathbb{P}(X + t\Delta X, S + \Delta S)\|_F^2$.
6. Update: $X_{k+1} = X_k + t\Delta X_k, S_{k+1} = S_k + t\Delta S_k$.

7. Update: $X_{k,1} = X_k + t\Delta X_k, S_{k,1} = S_k + t\Delta S_k$.
8. Solve $\mathbb{DP}_{(X,S)}(\Delta X_{k,1}, \Delta S_{k,1}) = -(\mathbb{P}(X_{k,1}, S_{k,1}), 0)$.
9. Update $X_{k+1} = X_{k,1} + \Delta X_{k,1}, S_{k+1} = S_{k,1} + \Delta S_{k,1}$ and go to 2.

Numerical experiments are based on test polynomials taken from the NLEVP library [1] and use starting points given either by a few steps of subspace iteration, or by a random choice (when the leading coefficient is singular).

Problem	**Deg. $\mathbb{P}$**	**Size X**	**Ite.**	**Time**
bicycle	2	2x2	11	0.067
butterfly	4	64x5	25	3.736
cd_player	2	60x6	6	0.904
closed_loop	2	2x2	4	0.056
damped_beam	2	200x5	7	3.516
dirac	2	80x5	36	5.422
hospital	2	24x24	110	22.191
metal_strip	2	9x9	19	0.663
mobile_manipulator	2	5x2	6	0.03
pdde_stability	2	225x6	4	3.185
planar_waveguide	4	129x5	43	16.674
plasma_drift	3	128x6	23	9.255
power_plant	2	8x8	33	3.11
railtrack	2	1005x3	25	216.53

Counting eigenvalues inside a contour. Being able to count or estimate the number of eigenvalues contained in a given contour Γ can of course be useful for computations based on a contour interval approach. We have tried experimenting with techniques suggested in [4].

3 Symbolic/numeric approach

In this part of our work we experiment with more symbolic and numeric approaches that can help to efficiently compute (a good approximation of) a required invariant pair, and/or provide a starting point for an iterative refinement method such as Newton.

Triangularization. Motivated by the results proposed in [7] and [6], we explore the role of triangularization in the computation of invariant pairs. Triangularization of a polynomial $P(\lambda)$ consists in determining an upper triangular matrix polynomial $T(\lambda)$ which is equivalent to $P(\lambda)$ and has the same degree. We perform this task via linearization: for instance, given a monic quadratic polynomial $A(\lambda) = \lambda^2 + A_1\lambda + A_0$, we consider its companion linearization $A = \begin{bmatrix} I & -A_1 \\ 0 & -A_0 \end{bmatrix}$ and compute an invertible matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$ such that $MAM^{-1} = T$, where $T = \begin{bmatrix} I & -T_1 \\ 0 & T_0 \end{bmatrix}$ is a linearization of a monic quadratic polynomial with upper triangular coefficients. Here the blocks M_{11}, M_{12} and M_{22} are unitary. The triangular form will prove useful for computing invariant pairs for $T(\lambda)$, particularly in the case where the blocks A_0 and A_1 are structured. Once invariant pairs

for $T(\lambda)$ are known, we can recover corresponding invariant pairs for $A(\lambda)$.

Subspace iteration. Subspace (orthogonal) iteration applied to a linearized form of $P(\lambda)$ may prove to be an efficient way to compute an approximated invariant pair to be refined later, particularly in the case where we look for a cluster of eigenvalues that are well separated from the rest of the spectrum.

References

[1] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. *http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html.*

[2] T. Betcke, D. Kressner. Perturbation, extraction and refinement of invariant pairs for matrix polynomials. *Linear Algebra Appl.*, 435:514–536, 2011.

[3] W.J. Beyn, V. Thümmler. Continuation of invariant subspaces for parameterized quadratic eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 31(3):1361–1381, 2009.

[4] Y. Futamura, Y. Maeda, T. Sakurai. Stochastic estimation method of eigenvalue density for nonlinear eigenvalue problem on the complex plane. *SIAM Letters, 3:61–64, 2011.*

[5] I. Gohberg, P. Lancaster, L. Rodman. Matrix Polynomials. Academic Press Inc., Computer Science and Applied Mathematics*, New York, 1982.*

[6] L. Taslaman, F. Tisseur, I. Zaballa. Triangularizing matrix polynomials. Linear Algebra Appl. *439(7), 1679–1699, 2013.*

[7] F. Tisseur, I. Zaballa. Triangularizing quadratic matrix polynomials. SIAM J. Matrix Anal. Appl.*, 34(2), 312–337, 2013.*

Backward error and conditioning of Fiedler linearizations

Fernando De Terán and Françoise Tisseur

The family of Fiedler pencils is a class of linearizations of matrix polynomials with many interesting features, from both the theoretical and the practical point of view. In particular, Fiedler pencils are *companion forms*, namely, templates which are easily constructible from the coefficients A_i of the matrix polynomial $P(\lambda)$, without performing any arithmetic operation. These are other useful properties of Fiedler pencils:

(a) They are always strong linearizations, regardless of $P(\lambda)$ being regular or singular (though, in this talk, we focus on *regular* polynomials).

(b) The left and right eigenvectors of $P(\lambda)$ are easily recovered from those of the companion form.

(c) The leading coefficient of the companion form is block diagonal, thereby reducing the computational cost of the Hessenberg-triangular reduction step of the QZ algorithm.

(d) They can be easily transformed into a block upper-triangular form revealing zero and infinite eigenvalues, if any.

(e) They are the source of many structured linearizations (like symmetric or palindromic).

In this talk, we analyze some numerical features of Fiedler linearizations. When a Polynomial Eigenvalue Problem (PEP) is solved by linearization, it is transformed into a Generalized Eigenvalue Problem, with different conditioning and backward error than the PEP. Hence, in order to measure the numerical performance of particular linearizations it is important to compare both the backward error and the conditioning of the linearization with those of the polynomial.

We will first display explicit formulas for the eigenvectors of any Fiedler pencil $F(\lambda)$ which, in particular, will show that the eigenvectors of $P(\lambda)$ can be easily recovered from those of the Fiedler pencil. From these formulas we get expressions for the condition number of eigenvalues of any Fiedler linearization, which allows us to compare this conditioning with the conditioning of the original polynomial, and to derive bounds for the ratio between these two condition numbers. These bounds depend on the norm of the coefficients A_i, together with the size and the degree of the polynomial.

Then, we will show expressions for the backward error of approximate eigenpairs of any Fiedler linearization $F(\lambda)$, and compare the backward error of approximate eigenpairs of $F(\lambda)$ with the backward error of the corresponding eigenpair of $P(\lambda)$. We will show bounds for the ratio between these two backward errors as well. We will see, in particular, that if the matrix polynomial is well scaled (i.e., $\|A_i\|_2 \approx 1$, for all $i = 0, 1, \ldots, k$), then all Fielder companion linearizations have good conditioning and backward stability properties.

We also consider diagonal scaling of the Fiedler pencils, and we show that there is always an appropriate scaling such that the ratio between the condition number of the eigenvalues in any two Fiedler pencils is bounded by some constant depending only on the size and the degree of the polynomial.

Some numerical experiments will be shown to illustrate our theoretical results.

The general inverse matrix polynomial index structure problem

Fernando De Terán, Froilán M. Dopico and Paul Van Dooren

Very recently, a fundamental result on matrix polynomials, baptized as the *Index Sum Theorem*, has been recovered from the literature in [1] and extended to arbitrary fields. Although this result was introduced in 1991 (see [2]) for matrix polynomials with real coefficients, it has remained unnoticed for many researchers. Given any matrix polynomial $P(\lambda)$, regular or singular, square or rectangular, with coefficients in an arbitrary field, the Index Sum Theorem establishes that if $\delta_{\text{fin}}(P)$ is the sum of the degrees of the finite elementary divisors of $P(\lambda)$, $\delta_{\text{inf}}(P)$ is the sum of the degrees of the infinite elementary divisors of $P(\lambda)$, and $\mu(P)$ is de sum of the left and right minimal indices of $P(\lambda)$, then

$$\delta_{\text{fin}}(P) + \delta_{\infty}(P) + \mu(P) = \text{degree}(P) \cdot \text{rank}(P). \tag{1}$$

Apart from its fundamental nature, it has been shown [1] that the Index Sum Theorem is a useful tool in many problems concerning matrix polynomials as, for instance, in the study of structure preserving linearizations.

The Index Sum Theorem immediately poses a natural inverse problem. Namely, given any list $\mathcal{L}$ of elementary divisors and left and right minimal indices such that

$$\delta_{\text{fin}}(\mathcal{L}) + \delta_{\infty}(\mathcal{L}) + \mu(\mathcal{L}) = r \cdot d \tag{2}$$

for two positive integers r and d, it is natural to ask if there exists a matrix polynomial of rank r and exact degree d with precisely these elementary divisors and left and right minimal indices.

We prove that the answer to this question is positive under the following two additional necessary and sufficient conditions: (1) the length of the longest chain of elementary divisors corresponding to the same eigenvalue in $\mathcal{L}$ is smaller than or equal to r; and (2) the length of the chain of infinite elementary divisors in $\mathcal{L}$ is strictly smaller than r.

Another important idea introduced in [1] is the concept of ℓ-*ification*, which extends to arbitrary degree ℓ the widely used notion of *linearization*. We show how the new inverse problem we have solved can be used to determine which are the possible sizes of ℓ-ifications and strong ℓ-ifications of a given matrix polynomial $P(\lambda)$.

Finally, we will discuss other inverse problems in matrix polynomials closely related to the one mentioned above.

References

[1] F. De Terán, F. M. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the index sum theorem. *Manchester Institute for Mathematical Sciences, The University of Manchester,* MIMS EPrint: 2013.47.

[2] C. Praagman. Invariants of polynomial matrices. Proceedings of the First European Control Conference, Grenoble 1991. (I. Landau, Ed.) INRIA, 1274-1277, 1991.

Rational operator functions and applications: the history of a symmetric matrix

Christian Engström
Umeå University, Sweden

Nonlinear matrix eigenvalue problems often have their origin in the discetization of a differential or integral operator. The properties, e.g. symmetries, clustering of eigenvalues, and condition numbers, of the resulting matrix eigenvalue problem depend on the underlying operators and the discretisation process. In recent years, it has been an increased interest in numerical analysis of self-adjoint spectral problems with non-empty essential spectrum. Points in the discrete spectrum, which are enclosed in between two points of the essential spectrum are called *non-variational* eigenvalues since the classical min-max principle only apply below the minimum of the essential spectrum. The non-variational eigenvalues can be approximated with the second-order projection method, which result in a non-symmetric eigenvalue problem [5]. However, variational principles can in some cases also be established above the minimum of the essential spectrum and we will consider such a case.

Let $A : \mathcal{H} \to \mathcal{H}$, with domain $\operatorname{dom} A$ denote a self-adjoint operator and let $B_\ell : \widehat{\mathcal{H}}_\ell \to \mathcal{H}$, $\ell = 1, 2, \ldots, L$ denote bounded linear operators between infinite dimensional Hilbert spaces. In this talk we consider rational operator functions in a Hilbert space $\mathcal{H}$ of the form

$$\mathcal{S}(\lambda) = A - \lambda - \sum_{\ell=1}^{L} \frac{B_\ell B_\ell^*}{c_\ell - \lambda}, \quad \operatorname{dom} \mathcal{S}(\lambda) = \operatorname{dom} A, \quad \lambda \in \mathbb{C} \setminus \{c_1, c_2, \ldots, c_L\}, \tag{1}$$

with $c_1 < c_2 < \cdots < c_L$. Moreover, we assume that A is semi-bounded from below and its resolvent is compact. The spectrum of $\mathcal{S}$ is then real and the operators $\mathcal{S}(\lambda)$, $\lambda \in \mathbb{C} \setminus \{c_1, c_2, \ldots, c_L\}$ have discrete spectrum [4]. If we set $\widehat{\mathcal{H}} := \widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_L$, then $\mathcal{S}$ is the Schur complement of the operator matrix $\mathcal{A}$ in the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \widehat{\mathcal{H}}$ given by

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 & \cdots & B_L \\ B_1^* & c_1 & 0 & \cdots & 0 \\ B_2^* & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_L^* & 0 & 0 & \cdots & c_L \end{pmatrix}, \quad \operatorname{dom} \mathcal{A} = \operatorname{dom} A \oplus \widehat{\mathcal{H}}. \tag{2}$$

The block operator matrix $\mathcal{A}$ is self-adjoint and bounded from below with essential spectrum $\{c_1, c_2, \ldots, c_L\}$ [1, 7]. Hence, we can apply the classical variational characterisation of the eigenvalues below c_1 [6]. In this talk, we establish variational principles in $(c_\ell, c_{\ell+1})$ without postulating that the infimum of the Rayleigh functional exists [4]. Moreover, we apply the new theory to an unbounded operator function with periodic coefficients. The main applications for this operator function are photonic crystals and periodic waveguides. These nano-sized structures can be used to control the flow of light and are for example used in integrated optics. The rational terms in the function (1) is in this application a consequence of frequency dependent material parameters [3, 2].

The block operator matrix $\mathcal{A}$ is discretised with a continuous and a discontinues high order finite element method and the advantages/disadvantages of these two methods are discussed. The discetization process results in a symmetric matrix eigenvalue problem and several examples illustrate the general theory developed in [4]. In particular, we show the connection between eigenvalue accumulation at c_ℓ and a numerical approximation of the corresponding singular sequence.

This talk is based on a joint work with Heinz Langer and Christiane Tretter.

References

[1] Atkinson, F. V., Langer, H., Mennicken, R., and Shkalikov, A. A., *The essential spectrum of some matrix operators*, Math. Nachr. **167**, 5–20 (1994).

[2] Effenberg, C., Kressner, D., and Engström, C., *Linearization techniques for band structure calculations in absorbing photonic crystals*, Int. J. Num. Meth. Eng. **89**, 180–191 (2012).

[3] Engström, C., *On the spectrum of a holomorphic operator-valued function with applications to absorptive photonic crystals*, Math. Models Methods Appl. Sci. **20**, 1319–1341 (2010).

[4] Engström, C., Langer, H., and Tretter, C., *Nonlinear eigenvalue problems and applications to photonics crystals: spectral theory and numerics*, In preparation.

[5] Levitin, M. and Shargorodsky, E., *Spectral pollution and second order relative spectra for self-adjoint operators*, IMA J. Numer. Anal., 24, 393416 (2004).

[6] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics, Volume 4: Analysis of Operators*, New York: Academic Press (1978).

[7] Tretter, C., *Spectral theory of block operator matrices and applications*, Imperial College Press (2008).

Solving matrix nearness problems using the implicit determinant method

Melina A Freitag and Alastair Spence

We provide algorithms for the solution of several matrix nearness problems:

- The distance of a stable matrix to the unstable matrices [1]. The method is based on finding a two-dimensional Jordan block corresponding to a pure imaginary eigenvalue in a certain two-parameter Hamiltonian eigenvalue problem introduced in [2].
- The computation of the distance of a matrix to a nearby defective matrix [3]. The problem is formulated following Alam & Bora [4] and reduces to finding when a parameter-dependent matrix is singular subject to a constraint.
- The calculation of the real stability radius [5], which uses a formula introduced by Qiu et al [6].
- The computation of Jordan blocks in parameter-dependent matrices [7], which has important applications in many physical problems.
- The calculation of the H_∞-norm [8].

The solution to all these problems is achieved by extensions and variations of the Implicit Determinant Method [9]. Numerical results show the performance of the algorithms for several examples and comparison is made with other methods for the same problem.

References

[1] M. A. Freitag and A. Spence, *A Newton-based method for the calculation of the distance to instability*, Linear Algebra Appl., 435 (2011), pp. 3189 – 3205.

[2] R. Byers, *A bisection method for measuring the distance of a stable matrix to the unstable matrices*, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 875–881.

[3] R. O. Akinola, M. A. Freitag, and A. Spence, *The calculation of the distance to a nearby defective matrix.*, (2013). to appear in Numer. Linear Algebra Appl. doi: 10.1002/nla.1888

[4] R. Alam and S. Bora, *On sensitivity of eigenvalues and eigendecompositions of matrices*, Linear Algebra Appl., 396 (2005), pp. 273–301.

[5] M. A. Freitag and A. Spence, *A new approach for calculating the real stability radius*, (2013). to appear in BIT Numerical Mathematics. doi: 10.1007/s10543-013-0457-x

[6] L. Qiu, B. Bernhardsson, A. Rantzer, E. Davison, P. Young, and J. Doyle, *A formula for computation of the real stability radius*, Automatica, 31 (1995), pp. 879–890.

[7] M. A. Freitag and A. Spence, *A method for the computation of Jordan blocks in parameter-dependent matrices*, (2013). to appear in IMA Journal of Numerical Analysis. doi: 10.1093/imanum/drt028

[8] M. A. Freitag, A. Spence, and P. V. Dooren, *Calculating the H_∞-norm using the implicit determinant method*, (2014). to appear in SIAM J. Matrix Anal. Appl.

[9] A. Spence and C. Poulton, *Photonic band structure calculations using nonlinear eigenvalue techniques*, J. Comput. Phys., 204 (2005), pp. 65–81.

Accepting a Root of $\det(A(\beta)) = 0$ and Computing the Corresponding Vector $x(\beta)$

Sven Hammarling*

1 Introduction

This is an historical contribution based upon an Appendix written for Ferriss et al. [1983], in which we needed to solve nonlinear eigenvalue problems, where the non-zero elements of the n by n matrix A are nonlinear functions of a scalar λ. At that time we still followed the Wilkinson tradition of finding the eigenvalues as the roots of the equation $\det(A(\lambda)) = 0$, for example [Wilkinson, 1965, Chapter 7, Section 63]. Having found a root, we wished to verify that it was acceptable as an eigenvalue and to then compute a corresponding eigenvector.

A zero, β, of the determinant of the matrix $A(\lambda)$ was computed by an NPL root finder that used Muller's method, or three point rational interpolation, [NPL, 1984]. The routine allowed for very large or small values, which is necessary when computing determinants, see for example [Peters and Wilkinson, 1970, Section 9] and [Wilkinson and Reinsch, 1971, Contribution I/7].

Apart from some minor tidying and LaTeXing, the Appendix is as originally written.

2 The Appendix

Let β be a computed zero of the determinant of the matrix $A(\lambda)$, put

$$A = A(\beta)$$

and let the singular value decomposition of A be

$$A = Q\Sigma P^T, \quad Q^TQ = P^TP = I, \quad \Sigma = \text{diag}(\sigma_i),$$

with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

If β has been computed reliably then it will be the exact eigenvalue of a neighbouring matrix, say B, so that

$$B = A + E, \quad \|E\| \leq \epsilon \|A\|, \ \epsilon \text{ small}, \quad \det(B) = 0.$$

*School of Mathematics, The University of Manchester. www.maths.manchester.ac.uk/~sven/

We can expect ϵ to be approximately the relative accuracy in the matrix $A(\lambda)$ in the neighbourhood of β. Let the rank of B be r where, since B is exactly singular, r is less than n. Then we have [Wilkinson, 1978]

$$\|E\|_E \geq (\sigma_{r+1}^2 + \sigma_{r+2}^2 + \cdots + \sigma_n^2)^{\frac{1}{2}},$$

and hence

$$(\sigma_{r+1}^2 + \sigma_{r+2}^2 + \cdots + \sigma_n^2)^{\frac{1}{2}} \leq \epsilon \|A\|_E \leq n^{\frac{1}{2}} \epsilon \|A\|_2 = n^{\frac{1}{2}} \epsilon \sigma_1,$$

where $\|A\|_E$ denotes the Euclidean (or Frobenius) norm of A and $\|A\|_2$ denotes the spectral norm of A.

Thus, if β is a reliably computed root of $\det(A) = 0$ we can certainly expect that

$$\sigma_n \leq \mathit{tol} \,.\, \sigma_1, \tag{1}$$

where *tol* is the relative accuracy in the elements of A. This gives us a reliable test for acceptance of an alleged root of $\det(A(\beta)) = 0$.

Note that if β is a computed root of multiplicity $(n - r)$ then we expect that

$$\sigma_i \leq \mathit{tol} \,.\, \sigma_1, \ i = r+1, r+2, \ldots, n.$$

Now let us suppose that β is an *exact* root of $A(\lambda)$ of multiplicity $(n - r)$. If x is a corresponding vector we put

$$y = P^T x,$$

and partition Σ and y as

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \ \ y = \begin{pmatrix} \tilde{y} \\ \hat{y} \end{pmatrix},$$

where D is r by r, so that D is non-singular, and $\tilde{y}$ is r by 1. Then, since x is to satisfy

$$Ax = 0, \ \ x \neq 0,$$

we have

$$Q \Sigma P^T x = 0$$

and hence

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \hat{y} \end{pmatrix} = 0.$$

This gives

$$\tilde{y} = 0, \ \ \hat{y} \neq 0, \text{ but otherwise arbitrary.}$$

Thus, we can take

$$y = e_j, \ \ j > r,$$

where e_j is the jth column of the unit matrix, which gives

$$x = p_j, \ \ j > r,$$

where p_j is the jth column of P.

Hence columns $p_{r+1}, p_{r+2}, \ldots, p_n$ gives an orthogonal set of vectors corresponding to a root β of multiplicity $(n-r)$.

In practice, therfore, we can compute the singular values, σ_i, and the right-hand singular vectors, p_i, of the singular value decomposition of A, use the test of equation (1) to determine r and, provided that $r < n$, accept β as a root and $p_{r+1}, p_{r+2}, \ldots, p_n$ as vectors. This computation will not normally be expensive relative to the cost of determining β.

Note that if A is symmetric then we can use the spectral factorization of A given by

$$A = PSP^T, \quad P^TP = I, \quad S = \text{diag}(s_i),$$

with

$$|s_1| \geq |s_2| \geq \cdots \geq |s_n| \geq 0, \text{ for which}$$
$$|s_i| = \sigma_i \text{ and } q_j = \text{sign}\,(s_j)\, p_j,$$

where q_j is the jth column of Q and

$$\text{sign}\,(a) = \begin{cases} +1, & a \geq 0 \\ -1, & a < 0. \end{cases}$$

References

D. H. Ferriss, S. Hammarling, D. W. Martin, and A. G. P. Warham. Numerical solution of equations describing electromagnetic propogation in dielectric tube waveguides. Technical Report DITC 16/83, National Physical Laboratory, Teddington, Middlesex TW11 0LW, UK, 1983.

NPL. *A Brief Guide to the NPL Linear Algebra Subroutine Library*. National Physical Laboratory, Teddington, Middlesex TW11 0LW, UK, 1984.

G. Peters and J. H. Wilkinson. $Ax = \lambda Bx$ and the generalized eigenproblem. *SIAM J. Num. Anal.*, 7:479–492, 1970.

J. H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, Oxford, UK, 1965.

J. H. Wilkinson. Singular-value decomposition – basic aspects. In D. A. H. Jacobs, editor, *Numerical Software – Needs and Availability*, pages 109–135. Academic Press, London, UK, 1978. (Proceedings of the IMA Conference, University of Sussex, 1977).

J. H. Wilkinson and C. Reinsch, editors. *Handbook for Automatic Computation, Vol.2, Linear Algebra*. Springer-Verlag, Berlin, Germany, 1971.

The infinite Arnoldi method and implicit restarting for nonlinear eigenvalue problems with low rank

Elias Jarlebring, Roel Vanbeeumen, Wim Michiels and Karl Meerbergen

We consider the nonlinear eigenvalue problem to find $(\lambda, v) \in \Omega \times \mathbb{C}^n \backslash 0$ such that

$$M(\lambda)v = 0,$$

where M is analytic in $\Omega \subset \mathbb{C}$, which is a neighborhood of the origin. In several applications, $M(\lambda)$ has a structure where the higher-order terms of the Taylor expansion of $M(\lambda)$ have low rank. More precisely, we will assume that M satisfies

$$M^{(i)}(0) = V_i Q^*, \quad \text{for } i \geq p+1,$$

for some $p > 0$ where $Q \in \mathbb{C}^{n \times r}$ is an orthogonal matrix and $V_i \in \mathbb{C}^{n \times r}$ for all $i = p+1, p+2, \ldots$. We propose a new algorithm that can exploit this structure when $r \ll n$. The algorithm is based on Arnoldi's method. More precisely, the algorithm is equivalent to Arnoldi's method applied to an operator whose reciprocal eigenvalues are solutions to the NEP. The algorithm can be interpreted as Arnoldi's method on this operator where the iterates are functions represented in a particular structured vector-valued polynomial basis. The construction is based on a similar construction in the infinite Arnoldi method [1]. Unlike [1] the low-rank structure allows us to consider a different operator leading to a structure exploitation that reduces the computational cost in particular the cost associated with orthogonalization. We maintain an equivalence with an Arnoldi method on an operator, and the structure exploitation provides a natural way to carry out implicit restarting and locking, with a Krylov-Schur approach similar to [2]. The efficiency and properties of the algorithm are illustrated with large-scale problems.

References

[1] E. Jarlebring, W. Michiels, and K. Meerbergen. A linear eigenvalue algorithm for the nonlinear eigenvalue problem. *Numer. Math.*, 122(1):169–195, 2012.

[2] G. W. Stewart. A Krylov–Schur algorithm for large eigenproblems. *SIAM J. Matrix Anal. Appl.*, 23(3):601–614, 2001.

Solvability Conditions for Undamped Gyroscopic Model Updating Problem

Zhigang Jia, Delin Chu, and Françoise Tisseur

Abstract

Model updating with no-spill over for a quadratic matrix polynomial

$$G(\lambda) = M\lambda^2 + C\lambda + K,$$

refers to updating $G(\lambda)$ so that a set of k measured eigenvalues and eigenvectors from a real-life problem replaces a set of k eigenpairs of $G(\lambda)$ known from computation, the other eigenpairs of $G(\lambda)$ remaining unchanged (i.e., no spill-over), and the physical and structural properties of $G(\lambda)$ are preserved such as positive definiteness of M and K.

Recently many approaches for this problem have been proposed, but these are mainly for quadratics with symmetric mass, damping and stiffness matrices M, C, K, respectively (see for example [1] and [3]). Our interest is in gyroscopic quadratics for which M and K are symmetric positive definite and the gyroscopic matrix C is skew-symmetric. Such quadratics play an important role in the design and analysis of vibrating structures including motors and aircrafts.

We present necessary and sufficient solvability conditions for the undamped gyroscopic model updating problem and derive an expression of the solution when it exists. This is done in two steps. First, a general updating formula with a parameter matrix Φ is given to fulfill the requirement of no spill-over, then an expression for Φ is derived that satisfies the other updating conditions, that is, the (skew-)symmetry of the gyroscopic matrix and the positive definiteness of the updated mass and stiffness matrices. We note that the unchanged $2n - k$ eigenpairs do not need to be known as they are not needed in the updating process.

Undamped gyroscopic quadratics with $M > 0$ and $K > 0$ have the property that their eigenvalues are pure imaginary and semisimple. These eigenvalues have a sign $+1$ or -1 attached to them forming the *sign characteristic* of $G(\lambda)$ [2]. To be more specific, the sign characteristic of a semisimple purely imaginary eigenvalue λ of $G(\lambda)$ with right eigenvector v is given by

$$\epsilon(\lambda) := \mathtt{sign}(-iv^*G^{(1)}(\lambda)v),$$

where $G^{(1)}(\lambda) = 2M\lambda + C$. The sign characteristic is an intrinsic property of undamped gyroscopic systems. It has a strong relation with the positive definiteness of the mass matrix M and the stiffness matrix K. Using the sign characteristic we present a necessary condition on the solvability of the undamped gyroscopic model updating problem.

References

[1] D. Chu, M. T. Chu, and W.-W. Lin. Quadratic model updating with symmetry, positive definiteness, and no spill-over. *SIAM J. Matrix Anal. Appl.*, 31:546–564, 2009.

[2] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications.* Birkhäuser, Basel, Switzerland, 2005.

[3] P. Lancaster. Model updating for self-adjoint quadratic eigenvalue problems. *Linear Algebra Appl.*, 428:2778–2790, 2008.

Algorithms for Hessenberg–Triangular Reduction of Fiedler Linearization of Matrix Polynomials

Lars Karlsson and Françoise Tisseur

Consider a regular complex matrix polynomial

$$P(\lambda) = \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0$$

of degree d where $P_k \in \mathbb{C}^{n\times n}$, $P_d \neq 0$, and $\det(P(\lambda))$ is not identically zero. The *polynomial eigenvalue problem* consists of finding complex scalars λ and complex nonzero vectors x and y such that $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. The vectors x and y are right respectively left eigenvectors associated with the eigenvalue λ.

The polynomial eigenvalue problem can be solved by constructing a linearization

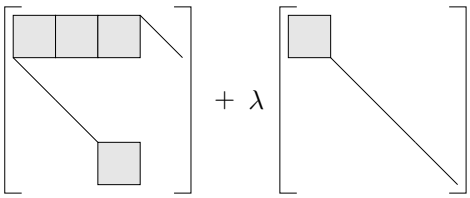
$$L(\lambda) = A + \lambda B, \quad A, B \in \mathbb{C}^{dn\times dn}$$

of $P(\lambda)$. For small- to medium-sized problems, the resulting generalized eigenvalue problem for $L(\lambda)$ can be solved using standard dense algorithms based on the QZ algorithm. However, before executing the QZ algorithm, the matrix pair (A, B) needs to be reduced to *Hessenberg–triangular (HT) form*, i.e., unitary matrices $Q, Z \in \mathbb{C}^{dn\times dn}$ needs to be found such that $H = Q^*AZ$ is upper Hessenberg and $T = Q^*BZ$ is upper triangular. The reduction to HT form is an expensive step in the solution process and hence parallel computation becomes necessary in order to solve larger problems in a reasonable amount of time.

There already exists a variety of efficient HT reduction algorithms for general unstructured inputs [2, 4]. But many linearizations have additional structure that can potentially be exploited to accelerate the HT reduction process. We have recently designed novel parallel algorithms for HT reduction of a particular Fiedler linearization [1, 3]. For $d = 4$, the linearization is given by

$$L(\lambda) = A + \lambda B = \begin{bmatrix} P_3 & P_2 & P_1 & -I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & P_0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

The sparsity structure of $L(\lambda)$ takes the pictorial form

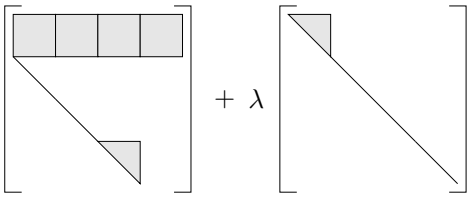


Note that the matrix pair is already almost in HT form with A in block Hessenberg form and B in block triangular form. The matrix B can be reduced to proper triangular form by computing a QR decomposition $P_d = Q_d R_d$ and pre-multiplying the first block row by Q_d^*. In addition, the $(d, d-1)$ block of A can also be reduced to triangular form by computing a QR decomposition $P_0 = Q_0 R_0$ followed by pre-multiplication of the last block row by Q_0^* and post-multiplication of the last block

column by Q_0. The transformed pencil $\tilde{L}(\lambda)$ takes the form

$$\tilde{L}(\lambda) = \tilde{A} + \lambda\tilde{B} = \begin{bmatrix} Q_4^*P_3 & Q_4^*P_2 & Q_4^*P_1 & -Q_4^*Q_0 \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & R_0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} R_4 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and has the pictorial sparsity structure



With $\tilde{B}$ in proper triangular form, band reduction algorithms based on a bulge-chasing technique can be applied to complete the HT reduction process efficiently (see, e.g., [2, 4, 5]). But the matrices $\tilde{A}$ and $\tilde{B}$ as well as the transformation matrices still have a rich structure. We have recently discovered that much of this structure is only gradually destroyed during the bulge-chasing process. Exploiting this phenomenon leads to faster algorithms.

We have developed a special-purpose parallel algorithm for multicore-based systems that is more than an order of magnitude faster than the general-purpose LAPACK implementation of HT reduction. Our algorithm combines previously published cache blocking techniques [4, 5] with a novel parallelization strategy. One of its features is that the task scheduling scheme promotes cache reuse by assigning tasks that operate on the same submatrix to the same thread. An analysis of execution profiles indicates that there is a potential to further improve the performance through automated tuning at runtime.

References

[1] E.N. Antoniou and S. Vologiannidis. A new family of companion forms of polynomial matrices. *Electron. J. Linear Algebra*, 11:78–87, 2004.

[2] K. Dackland and B. Kågström. Blocked algorithms and software for reduction of a regular matrix pair to generalized Schur form. *ACM Trans. Math. Software*, 25(4):425–546, 1999.

[3] F. De Terán, F.M. Dopico, and D.S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. SIAM J. Matrix Anal. Appl., 18:2121–2204, 2010.

[4] B. Kågström, D. Kressner, E.S. Quintana-Orti, and G. Quintana-Orti. Blocked algorithms for the reduction to Hessenberg–triangular form revisited. *BIT*, 48(3):563–584, 2008.

[5] B. Lang. Using level 3 BLAS in rotation-based algorithms. *SIAM J. Sci. Comput.*, 19(2):626–634, 1998.

Numerical observations on the Ritz values given by the incomplete orthogonalization method

Antti Koskela and Timo Eirola

In this work we show some interesting numerical observations on the eigenvalues and field of values of the Hesseberg matrix H_k which is obtained as a results of k steps of the *incomplete orthogonalization method* [1] applied to a matrix $A \in \mathbb{C}^{n\times n}$ and vector $b \in \mathbb{C}^n$.

In the incomplete orthogonalization method the new vector Aq_i is orthogonalized at step i only against m last basis vectors $\{q_{i-m+1}, \ldots, q_i\}$ instead of the whole basis as in the Arnoldi iteration. The coefficient h_{ij} are given by the inner products as in the Arnoldi iteration. The iteration is depicted by the following pseudocode.

Algorithm 1 The incomplete Arnoldi iteration for matrix A and starting vector b with an orthogonalization length m.

```
q_1 ← b/‖b‖
for i = 1, 2, ... do
  r ← Aq_i
  for j = max(1, i − m + 1) to i do
    h_{j,i} ← ⟨q_j, r⟩
    r ← r − h_{j,i} q_j
  end for
  h_{i+1,i} ← ‖r‖
  q_{i+1} ← r/h_{i+1,i}
end for
```

This gives the matrix

$$Q_k := \begin{bmatrix} q_1 & \ldots & q_k \end{bmatrix},$$

where the vectors q_i are orthogonal locally, i.e.,

$$\langle q_i, q_j \rangle = 0, \quad \text{if} \quad |i-j| \le m.$$

It is easy to verify that for the Krylov subspace $\mathcal{K}_k(A, b)$ holds that

$$\mathcal{K}_k(A, b) = R(Q_k),$$

where R denotes the range of a matrix. The iteration also gives the banded Hessenberg matrix

$$H_k = \begin{bmatrix} \langle Aq_1, q_1\rangle & \ldots & \langle Aq_m, q_1\rangle & & \\ \langle Aq_1, q_2\rangle & \ddots & \ddots & \ddots & \\ & \langle Aq_2, q_3\rangle & \ddots & \ddots & \langle Aq_{k-m+1}, q_k\rangle \\ & & \ddots & \ddots & \vdots \\ & & & \langle Aq_{k-1}, q_k\rangle & \langle Aq_k, q_k\rangle \end{bmatrix}.$$

Numerical experiments show interesting behaviour for the geometry of the spectrum $\Lambda(H_k)$ and field of values $\mathcal{F}(H_k)$.

In the following numerical example we consider a diagonal $A \in \mathbb{C}^{400\times 400}$ such that the diagonal elements are located in five circles as shown in figures below. Numerical experiments show that the geometry of $\Lambda(H_k)$ changes when the orthogonalization length m exceeds a multiple of 5 (the number of the circles). Here $k = 200$.

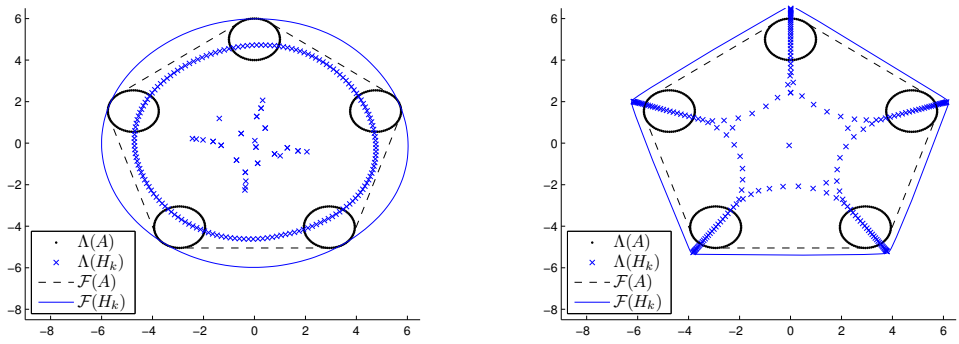


Figure 1: The spectra and fields of values for A and H_k when $m = 4$ and $m = 5$.

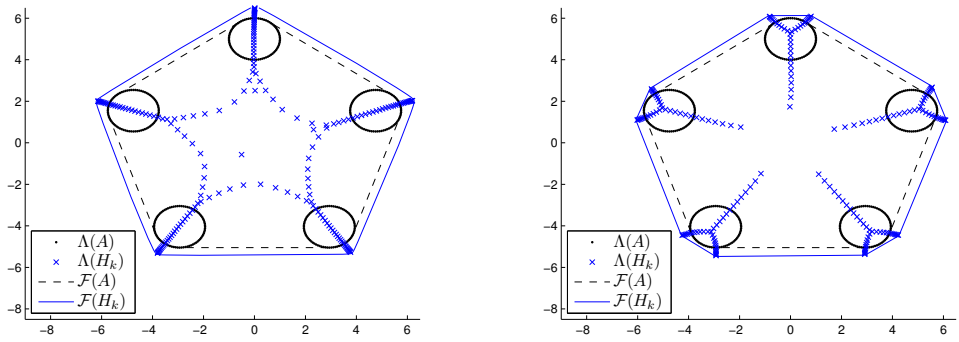


Figure 2: The spectra and fields of values for A and H_k when $m = 9$ and $m = 10$.

Similar behavior is observed also for other shapes of $\Lambda(A)$ (e.g., polygons, stars), and also for nonnormal matrices.

References

[1] Y. Saad. Variations on Arnoldi's method for computing eigenelements of large unsymmetric matrices. *Linear Algebr. Appl.* 34:269–295, 1980.

[2] A. Koskela and T. Eirola. Numerical observations on the Ritz values given by the incomplete orthogonalization method. *arXiv* (in preparation).

On the Residual Inverse Iteration for Nonlinear Eigenvalue Problems

Daniel Kressner
(joint work with Cedric Effenberger)

We consider nonlinear eigenvalue problems of the form

$$T(\lambda)x = 0, \qquad x \neq 0, \tag{1}$$

where $T : D \to \mathbb{C}^{n\times n}$ is a continuously differentiable matrix-valued function on some open interval $D \subset \mathbb{R}$.

In the following, $T(\lambda)$ is supposed to be Hermitian for every $\lambda \in D$. Moreover, we assume that the scalar nonlinear equation

$$x^* T(\lambda)x = 0 \tag{2}$$

admits a unique solution $\lambda \in D$ for every vector x in an open set $D_\rho \subset \mathbb{C}^n$. The resulting function $\rho : D_\rho \to D$, which maps x to the solution λ of (2), is called *Rayleigh functional*, for which we additionally assume that

$$x^* T'(\rho(x))x > 0 \qquad \forall x \in D_\rho.$$

The existence of such a Rayleigh functional entails a number of important properties for the eigenvalue problem (1), see [4, Sec. 115.2] for an overview. In particular, the eigenvalues in D are characterized by a min-max principle and thus admit a natural ordering. Specifically, if

$$\lambda_1 := \inf_{x\in D_\rho} \rho(x) \in D$$

then λ_1 is the first eigenvalue of T.

It is of interest to study the convergence of Neumaier's residual inverse iteration [3] for computing the eigenvalue λ_1 of T and an associated eigenvector x_1. In the Hermitian case, this iteration takes the form

$$v_{k+1} = \gamma_k(v_k + P^{-1}T(\rho(v_k))v_k), \quad k = 0, 1, \ldots, \tag{3}$$

with normalization coefficients $\gamma_k \in \mathbb{C}$, an initial guess $v_0 \in \mathbb{C}^n$, and a Hermitian preconditioner $P \in \mathbb{C}^{n\times n}$. Usually, $P = -T(\sigma)$ for some shift σ not too far away from λ_1 but the general formulation (3) allows for more flexibility, such as the use of multigrid preconditioners.

In [3, Sec. 3], it was shown that (3) with $P = T(\sigma)$ converges linearly to an eigenvector belonging to a simple eigenvalue, provided that σ is sufficiently close to that eigenvalue. Jarlebring and Michiels [2] derived explicit expressions for the convergence rate by viewing (3) as a fixed point iteration and considering the spectral radius of the fixed point iteration matrix.

Our new convergence analysis is tailored to the particular situation of having a Rayleigh functional, and differs significantly from [2, 3]. Our major motivation for reconsidering this question was to establish mesh-independent convergence rates when applying (3) with a multigrid preconditioner to the finite element discretization of a nonlinear PDE eigenvalue problem. The analyses in [2, 3] do not seem to admit such a conclusion, at least it is not obvious. On the other hand, such results

are well known for the linear case $T(\lambda) = \lambda I - A$, for which (3) comes down to the preconditioned inverse iteration (PINVIT). In particular, the seminal work by Neymeyr establishes tight expressions for the convergence of the eigenvalue and eigenvector approximations produced by PINVIT. Unfortunately, the elegance of Neymeyr's mini-dimensional analysis of the Rayleigh-quotient is strongly tied to linear eigenvalue problems; there seems little hope to carry it over to the general nonlinear case. Our approach proceeds by directly analysing the convergence of the eigenvector. Although leading to weaker bounds than Neymeyr's analysis in the linear case, the obtained results still allow to establish mesh-independent convergence rates.

In the first step, we show that

$$\tan \phi_P(v_{k+1}, x_1) \leq \gamma \cdot \tan \phi_P(v_k, x_1) + O(\varepsilon^2),$$

where x_1 is an eigenvector belonging to λ_1 and ϕ_P denotes the angle in the geometry induced by P. In the second step, we show that $\gamma < 1$ (independent of h) for a multigrid preconditioner of $T(\sigma)$ with σ sufficiently close to λ_1.

References

[1] Effenberger, C., Kressner, D.: On the residual inverse iteration for nonlinear eigenvalue problems admitting a Rayleigh functionals. Preprint available from `http://anchp.epfl.ch`, January 2014.

[2] Jarlebring, E., Michiels, W.: Analyzing the convergence factor of residual inverse iteration. BIT **51**(4), 937–957 (2011).

[3] Neumaier, A.: Residual inverse iteration for the nonlinear eigenvalue problem. SIAM J. Numer. Anal. **22**(5), 914–923 (1985)

[4] Voss, H.: Nonlinear eigenvalue problems. In: L. Hogben (ed.) Handbook of Linear Algebra. Chapman & Hall/CRC, FL (2013).

Linearizations for Interpolation Bases

Piers W. Lawrence

A standard approach to solving the polynomial eigenvalue problem is to linearize, which is to say the problem is transformed into an equivalent larger order generalized eigenproblem. For the monomial basis, much work has been done to show the conditions under which linearizations produce small backward errors, especially for the quadratic eigenvalue problem [3, 4]. Recently, there has been increasing interest in linearizations of polynomials expressed in bases other than the classical monomial basis [1]. In these bases, there is a need to establish the conditions under which linearizations return eigenvalue and eigenvector estimates with small backward errors.

In this work, we investigate the accuracy and stability of polynomial eigenvalue problems solved by linearization. The polynomial eigenvalue problems are expressed in the Lagrange basis, that is, by their values at distinct interpolation nodes. We also utilize the barycentric Lagrange formulation of the polynomial matrices, since the linearizations that arise from this formulation are particularly simple, and are flexible for computations. An m by m matrix polynomial $\mathbf{P}(\lambda)$ of degree n, expressed in the barycentric Lagrange formulation is

$$\mathbf{P}(\lambda) = \prod_{i=0}^{n}(\lambda - x_i)\sum_{j=0}^{n}\frac{w_j}{\lambda - x_j}\mathbf{F}_j\,, \quad w_j^{-1} = \prod_{k\neq j}(x_j - x_k)\,.$$

The numbers w_j are known as the barycentric weights, and the coefficients $\mathbf{F}_j = \mathbf{P}(x_j) \in \mathbb{C}^{m\times m}$ are the samples of $\mathbf{P}(\lambda)$ at the $n+1$ interpolation nodes x_j. An $(n+2)m$ by $(n+2)m$ linearization of the matrix polynomial $\mathbf{P}(\lambda)$ is given by [2]

$$\lambda\mathbf{B} - \mathbf{A} = \begin{bmatrix} \mathbf{0}_m & \mathbf{F}_0 & \dots & \mathbf{F}_n \\ -w_0\mathbf{I}_m & (\lambda - x_0)\mathbf{I}_m & & \\ \vdots & & \ddots & \\ -w_n\mathbf{I}_m & & & (\lambda - x_n)\mathbf{I}_m \end{bmatrix}. \tag{1}$$

This linearization introduces $2m$ spurious infinite eigenvalues. However, these spurious eigenvalues are not a problem in practice, and can be deflated from the pencil with little extra effort. The advantage of introducing the spurious eigenvalues comes in the form of flexibility when balancing the linearization. Furthermore, when the interpolation nodes are real or on the unit circle we may perform a reduction to block Hessenberg form in only $O(n^2)$ operations. This process is related to an inverse eigenvalue problem for computing the recurrence coefficients of orthogonal polynomials with respect to a discrete inner product defined by the interpolation nodes [5].

For the linearization (1), we show the conditions under which the backward error of the polynomial eigenvalue problem is not much larger than that of the backward error of the linearization. We also investigate the stability of two smaller linearizations recently proposed for polynomials expressed in barycentric form [6].

References

[1] A. Amiraslani, R. M. Corless, and P. Lancaster, *Linearization of matrix polynomials expressed in polynomial bases*, IMA Journal of Numerical Analysis, 29 (2009), pp. 141–157.

[2] R. M. Corless, *Generalized companion matrices in the Lagrange basis*, in Proceedings EACA, L. Gonzalez-Vega and T. Recio, eds., June 2004, pp. 317–322.

[3] S. Hammarling, C. J. Munro, and F. Tisseur, *An algorithm for the complete solution of quadratic eigenvalue problems*, ACM Trans. Math. Softw., 39 (2013), pp. 18:1–18:19.

[4] N. Higham, R. Li, and F. Tisseur, *Backward error of polynomial eigenproblems solved by linearization*, SIAM Journal on Matrix Analysis and Applications, 29 (2008), pp. 1218–1241.

[5] M. Van Barel and A. Bultheel, *Orthonormal polynomial vectors and least squares approximation for a discrete inner product*, ETNA, 3 (1995), pp. 1–23.

[6] R. Van Beeumen, W. Michiels, and K. Meerbergen, *Linearization of Lagrange and Hermite interpolating matrix polynomials*, tech. rep., TW Reports, TW627, Department of Computer Science, KU Leuven, 2013.

Quasi-Canonical Forms for Quadratic Matrix Polynomials

D. Steven Mackey, with F. De Terán, F. Dopico, Vasilije Perović, and Francoise Tisseur

Abstract

The Weierstrass and Kronecker canonical forms for matrix pencils [2] are indispensable tools for obtaining insight into both the theoretical and computational behavior of pencils and their corresponding eigenproblems. The absence of any analogous result for matrix polynomials of higher degree has made it difficult to achieve the same depth of understanding for general matrix polynomials as we have for pencils. In this talk I will describe recent progress towards canonical forms for *quadratic* matrix polynomials, both for general quadratics (square and rectangular) as well as for polynomials in various important structure classes such as Hermitian and palindromic quadratic matrix polynomials.

As a first step towards resolving the quadratic canonical form question, it is convenient to consider a given list $\mathcal{L}$ of elementary divisors (finite and/or infinite) together with (left and/or right) minimal indices as the initial data. Then the *quadratic realizability problem* (QRP) consists of two basic issues:

- *Characterize* those lists $\mathcal{L}$ that comprise the complete spectral and singular structure of some quadratic matrix polynomial.
- For each such list $\mathcal{L}$, show how to *concretely construct* a quadratic matrix polynomial that realizes the list $\mathcal{L}$. It is also desirable for this concrete realization to be as simple and canonical as possible.

Structured versions of the QRP can also be formulated — given a class $\mathcal{S}$ of structured matrix polynomials, characterize (and constructively realize) those lists $\mathcal{L}$ that comprise the complete spectral and singular structure of some quadratic matrix polynomial *in* $\mathcal{S}$. Thus we have the palindromic QRP, the alternating QRP, the Hermitian QRP, ..., etc.

Solutions to many of these QRPs have recently been obtained by developing *quasi-canonical forms* for both structured and unstructured quadratic matrix polynomials. These quasi-canonical forms are direct sums of canonical Kronecker-like quadratic blocks, each of which realizes an "irreducible" list of elementary divisors and minimal indices.

The development of these quasi-canonical forms has revealed several new phenomena that occur for quadratic polynomials, but not for matrix pencils.

(a) Although *any* list $\mathcal{L}$ can be realized by a matrix pencil (simply build a Kronecker block for each individual elementary divisor and for each minimal index), this is not true in the quadratic case. There are necessary conditions on $\mathcal{L}$ for quadratic realizability arising from the Index Sum Theorem [1]; for structured quadratic realizability there are additional necessary conditions arising from the restricted Smith forms of these structure classes [3, 4, 5].

(b) There exist nontrivial *quadratically irreducible* lists $\mathcal{L}$, i.e., quadratically realizable lists $\mathcal{L}$ with at least two elements that *cannot be partitioned* into quadratically realizable sublists. Clearly each quadratically irreducible list must be realized by its own individual block; the existence of such lists thus constitutes an important reason why the QRP is significantly more complicated than the corresponding realizability problem for pencils. Identifying all of the qualitatively distinct types of quadratically irreducible lists is one of the main contributions of this work.

(c) There exist nontrivial quadratically irreducible lists $\mathcal{L}$ that contain a mixture of both elementary divisors *and* minimal indices. The existence of such lists shows that it is not always possible to cleanly separate a quadratic matrix polynomial into a "regular part" and a "singular part", as can always be done for pencils.

(d) The canonical blocks in the classical Weierstrass and Kronecker forms for pencils are all bidiagonal. In the quadratic case, though, it can be shown that *bidiagonal blocks do not suffice*; i.e., there are quadratically irreducible lists $\mathcal{L}$ that are not realizable by any bidiagonal quadratic matrix polynomial. Thus the canonical blocks in any quadratic canonical form must necessarily be more complicated than those in the canonical forms for pencils.

The condensed quadratic realizations developed in this work are built as direct sums of quadratic blocks, each of which is canonical for some quadratically irreducible sublist of the given list $\mathcal{L}$ of elementary divisors and minimal indices. However, the quadratic condensed form as a whole is *not always unique*; there exist quadratically realizable lists $\mathcal{L}$ that can be partitioned into quadratically irreducible sublists in more than one way, leading to qualitatively distinct quadratic realizations. For this reason (and others to be discussed in the talk), the quadratic condensed forms presented here cannot be regarded as canonical forms in the usual sense of the term. Hence we refer to these condensed forms only as *quasi*-canonical forms, since they possess only some of the properties associated with a true canonical form.

References

[1] F. De Terán, F. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the index sum theorem. Available as MIMS EPrint 2013.47, Manchester Institute for Mathematical Sciences, Manchester, England, 2013.

[2] F. R. Gantmacher. *Theory of Matrices*. Chelsea, New York, 1959.

[3] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.

[4] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms of palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[5] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Skew-symmetric matrix polynomials and their Smith forms. *Linear Algebra Appl.*, 438(12):4625–4653, 2013.

Asymptotic Eigenvalue Problems

Marianne Akian, Stéphane Gaubert, James Hook, Andrea Marchesini and Françoise Tisseur

We use tropical algebra to study the behaviour of condition number and the accuracy of numerical computation of the eigenvalues of a matrix polynomial whose coefficient matrices are functions of a parameter ϵ.

Consider the complex matrices $a_0, \dots, a_d \in \mathbb{C}^{n\times n}$, and the tropical matrices $A_0, \dots, A_d \in \mathbb{R}_{\max}^{n\times n}$, where $\mathbb{R}_{\max} = \big(\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +\big)$. Akian, Bapat and Gaubert [ABG04] showed that if $\mathcal{A}_\epsilon$ is a matrix polynomial of the form

$$\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon,0} + \mathcal{A}_{\epsilon,1}\, x + \cdots + \mathcal{A}_{\epsilon,d}\, x^d \ ,$$

where the $\mathcal{A}_{\epsilon,k}$ are $n \times n$ matrices of the form $(\mathcal{A}_{\epsilon,k})_{ij} = (a_k)_{ij}\, \epsilon^{-(A_k)_{ij}} + o\big(\epsilon^{-(A_k)_{ij}}\big)$, and if γ is a finite tropical eigenvalue of the tropical pencil

$$A = A_0 \oplus A_1 x \oplus \cdots \oplus A_d x^d \tag{1}$$

with multiplicity m_γ, then $\mathcal{A}_\epsilon$ has generically m_γ eigenvalues $\mathcal{L}_{\epsilon,1}, \dots, \mathcal{L}_{\epsilon,m_\gamma}$ with asymptotics $\mathcal{L}_{\epsilon,i} \sim \lambda_i \epsilon^{-\gamma}$.

The numerical computation of such eigenvalues with some standard tool (e.g. MATLAB's `polyeig` function, or any QZ algorithm applied to a linearization of $\mathcal{A}_\epsilon$) is in general ill-conditioned and inaccurate for small values of ϵ, due in part to the fact that the entries of the coefficient matrices can be highly unbalanced. We propose some scaling methods to improve the conditioning of the eigenpairs for small values of ϵ.

One method consists in replacing (1) with the scaled tropical pencil $\tilde{A} = \tilde{A}_0 \oplus \tilde{A}_1 y \oplus \cdots \oplus \tilde{A}_d y^d$, with $\tilde{A}_k = A_k + k\alpha$ (obtained by the change of variable $x = \alpha \otimes y$), and changing $\mathcal{A}_\epsilon$ to $\tilde{\mathcal{A}}_\epsilon$ accordingly. The eigenvalues of $\mathcal{A}_\epsilon$ can be retrieved from those of $\tilde{\mathcal{A}}_\epsilon$ by multiplying the latters by $\epsilon^{-\alpha}$.

This scaling does not change the condition number of any eigenvalue Λ of a matrix polynomial ($\kappa_P(\Lambda, \mathcal{A}_\epsilon) = \kappa_P(\Lambda\epsilon^\alpha, \tilde{\mathcal{A}}_\epsilon)$). However, it does change the backward error $\beta_P(\Lambda, \mathcal{A}_\epsilon)$, as well as the condition number of Λ seen as an eigenvalue of an associated linearized pencil, $\kappa_L(\Lambda, \mathcal{A}_\epsilon)$.

The two latter quantities can be very large for small values of ϵ. If γ is a tropical eigenvalue of A, by choosing $\alpha = \gamma$ in the above scaling, one can in some cases improve both $\beta_P(\Lambda, \mathcal{A}_\epsilon)$ and $\kappa_L(\Lambda, \mathcal{A}_\epsilon)$ for the eigenvalues Λ of order $\epsilon^{-\gamma}$ of $\mathcal{A}_\epsilon$, but the same quantities might at the same time be deteriorated for other eigenvalues. This means that one has to apply the method with all distinct tropical eigenvalues of A to obtain the whole spectrum of $\mathcal{A}_\epsilon$. A similar scaling was already proposed using the tropical roots of the polynomial $\|A_0\| \oplus \|A_1\| x \oplus \cdots \oplus \|A_d\| x^d$ instead of the tropical eigenvalues of A (see [GS09, HMT13]).

Further improvement can be obtained by applying what we call the *hungarian method*. We interpret the tropical pencil A as being the value matrix associated to a parametric optimal assignment problem with parameter x: an *hungarian pair* (U, V) is then any pair of potentials (i.e. vectors of dual variables) obtained by using the Hungarian algorithm to solve this optimal assignment problem for $x = \alpha$. The hungarian method consists in scaling the coefficients of (1) as follows: $A_k \mapsto H_k$, $(H_k)_{ij} = (A_k)_{ij} + k\alpha - U_i - V_j$, and changing $\mathcal{A}_\epsilon$ to $\mathcal{H}_\epsilon$ accordingly. Again, the eigenvalues of $\mathcal{A}_\epsilon$ can be retrieved from those of $\mathcal{H}_\epsilon$ by multiplying the latters by $\epsilon^{-\alpha}$, and if we choose $\alpha = \gamma$, where γ is a tropical eigenvalue of A, we are guaranteed that the condition number

of the eigenvalues of order $\epsilon^{-\gamma}$ of $\mathcal{A}_\epsilon$ will be finite for every value of ϵ (which may not be true for the other eigenvalues). Numerical computations performed for different values of ϵ, on randomly generated matrix polynomials, confirm these results.

References

[ABG04] M. Akian, R. Bapat, and S. Gaubert. Perturbation of eigenvalues of matrix pencils and optimal assignment problem. *C. R. Acad. Sci. Paris, Série I*, 339:103–108, 2004.

[HMT13] S. Hammarling, C. J. Munro, and F. Tisseur. An algorithm for the complete solution of quadratic eigenvalue problems. *ACM Transactions on Mathematical Software (TOMS)*, 39(3), 2013.

[GS09] S. Gaubert and M. Sharify. Tropical scaling oaf polynomial matrices. In Rafael Bru and Sergio Romero-Vivó, editors, *Proceedings of the third Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 09)*, volume 389 of *LNCIS*, pages 291–303, Valencia, Spain, 2009. Springer. arXiv:arXiv:0905.0121.

Global and local search methods for solving nonlinear eigenvalue problems with unknown nonlinear functions

Karl Meerbergen, Wim Michiels and Roel Van Beeumen

Let

$$A(\lambda)x = 0 \tag{1}$$

be a nonlinear eigenvalue problem with $A : \mathbb{C} \to\in \mathbb{C}^{n\times n}$. In many cases, $A(\lambda)$ can be written in the form

$$A(\lambda) = \sum_{j=1}^{m} \phi_j(\lambda) A_j \tag{2}$$

where ϕ_j are scalar functions of λ and A_j are constant matrices. There are basically four approaches for computing a selection of eigenvalues of (1). The first class of methods is based on a local approximation in one or two points, namely the residual inverse iteration method and Rayleigh quotient iteration or Newton's method [6] [5]. Convergence is local and convergence to a specific eigenvalue may therefore be less reliable. Globally convergent methods have been proposed that approximate the nonlinear function by a (rational) polynomial and then solve a polynomial or rational eigenvalue problem through a linearization [2] [7]. The third class of methods lies somewhere in between: a local approximation is built and adapted at each iteration, which makes it more and more a global approximation. The latter techniques adaptively increase the degree of the polynomial [4][8][3]. The fourth class of methods are using contour integrals for computing all eigenvalues inside a contour [1].

Which of those techniques are appropriate for solving which eigenvalue problem? Are there missing algorithms or approximations, for general, or for specific problems? One particular situation is where $A(\lambda)$ is not explicitly available in the form (2), or where $A(\lambda)$ is available in tabled form for a discrete number of values of λ.

References

[1] W.-J. Beyn. An integral method for solving nonlinear eigenvalue problems. *Linear Alg. Appl.*, 436(10):3839–3863, 2012.

[2] C. Effenberger and D. Kressner. Chebyshev interpolation for nonlinear eigenvalue problems. *BIT*, 52(4):933–951, 2012.

[3] S. Güttel, R. Van Beeumen, K. Meerbergen, and W. Michiels. NLEIGS: a class of robust fully rational Krylov methods for nonlinear eigenvalue problems. Technical Report TW633, Department of Computer Science, KU Leuven, Celestijnenenlaan 200A, 3001 Leuven, Belgium, 2013.

[4] E. Jarlebring, K. Meerbergen, and W. Michiels. An Arnoldi like method for the delay eigenvalue problem. *SIAM J. Sci. Comput.*, 32(6):3278–3300, 2010.

[5] D. Kressner. A block Newton method for nonlinear eigenvalue problems. *Numerische Mathematik*, 114(2):355–372, 2009.

[6] A. Neumaier. Residual inverse iteration for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 22:914–923, 1998.

[7] Y. Su and Z. Bai. Solving rational eigenvalue problem via linearization. *SIAM Journal on Matrix Analysis and Applications*, 32(1):201–216, 2011.

[8] R. Van Beeumen, K. Meerbergen, and W. Michiels. A rational Krylov method based on Hermite interpolation for nonlinear eigenvalue problems. *SIAM J. Sci. Comput.*, 35(1):A327–A350, 2013.

Invariants of structured matrix polynomials

Christian Mehl

What are the invariants of regular structured matrix polynomials?

The answer to this question is well known if the term "structured" is disregarded. In that case, the Smith form can be used to display the elementary divisors of the matrix polynomial, or, in other words, its eigenvalues and multiplicities. Concerning the eigenvalue infinity, the Smith form of the reversal can be considered.

However, the story is different if additional symmetry structures are present. In that case, it may happen that additional invariants (e.g., sign characteristics) occur if the equivalence tranformations are restricted to structure-preserving transformations.

As an example, we consider the case of T-even pencils, i.e., matrix pencils of the form $\lambda E - A$, where $E \in \mathbb{F}^{n\times n}$ is skew-symmetric and $A \in \mathbb{F}^{n\times n}$ is symmetric. Here, it is well known that two matrix pencils $\lambda E_1 - A_1$ and $\lambda E_2 - A_2$ that have the same Smith form are actually strictly equivalent, i.e., there exist nonsingular matrices $P, Q \in \mathbb{F}^{n\times n}$ such that $PE_2Q = E_1$ and $PA_2Q = A_1$.

However, if P, Q are arbitrary nonsingular matrices, then in general PE_2Q and PA_2Q need not be skew-symmetric or symmetric, respectively. The corresponding structure preserving transformations are thus congruence transformations, i.e., strict equivalence transformations for which $Q = P^T$. The question now is whether additional invariants are introduced when the set of transformations is restricted to congruence instead of strict equivalence.

The answer is no if $\mathbb{F} = \mathbb{C}$, but yes if $\mathbb{F} = \mathbb{R}$. To see this, consider the following example. Let

$$\lambda E_1 - A_1 = \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \lambda E_2 - A_2 = \lambda \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then both pencils have the same eigenvalues $\pm i$ with algebraic multiplicities one, and are therefore strictly equivalent. If $\mathbb{F} = \mathbb{C}$ then they are also congruent which follows immediately from $PE_2P^T = E_1$ and $PA_2P^T = A_1$ for $P = iI_2$. However, if $\mathbb{F} = \mathbb{R}$, then the two pencils are not congruent as there exists no nonsingular matrix $P \in \mathbb{R}^{2\times 2}$ such that $PA_2P^T = A_1$ due to Sylvester's Law of Inertia.

If T-alternating matrix polynomials are considered, i.e., polynomials satisfying $P(\lambda)^T = \pm P(-\lambda)$, then unimodular equivalence transformations can be used to compute the Smith form of $P(\lambda)$. Those are transformations of the form $P(\lambda) \mapsto E(\lambda)P(\lambda)F(\lambda)$, where $E(\lambda)$ and $F(\lambda)$ are unimodular, i.e., their determinant is a nonzero constant. Structure-preserving transformation on the other hand would be *unimodular congruence transformations* of the form $P(\lambda) \mapsto E(\lambda)P(\lambda)E(-\lambda)^T$, where $E(\lambda)$ is unimodular. Again, we may ask the question whether additional invariants are introduced when we restrict ourselves to unimodular congruence.

In the talk, we will see why it is difficult to find a complete answer to this question for several cases of structured matrix polynomials. In fact, so far only the case of skew-symmetric matrix polynomials is well understood.

Eigenvector error bounds and perturbation for nonlinear eigenvalue problems

Yuji Nakatsukasa* and Françoise Tisseur†

A computed approximate eigenpair $(\widehat{\lambda}, \widehat{x})$ of a polynomial eigenvalue problem $P(\lambda)x = 0$ where $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, $A_i \in \mathbb{C}^{n\times n}$ is considered acceptable in practice if the residual $\|P(\widehat{\lambda})\widehat{x}\|$ is small, as this indicates the solution has small backward error [6]. The forward error in the eigenvalue $|\widehat{\lambda}-\lambda|$ can be bounded for example by bounding the error in $\widehat{\lambda}$ as an eigenvalue of a linearization $\lambda X + Y$ of $P(\lambda)$ via eigenvalue perturbation theory for linear eigenvalue problems [4]. For the eigenvector, however, no previous result seems to be known that rigorously bound the error in $\widehat{x}$. Previous studies exist on the sensitivity of eigenvectors of matrix polynomials under perturbation [1, 2], but these deal only with first-order perturbation analysis, leading to the condition number of the eigenvector.

In this work we derive a posteriori upper bounds for the angle $\angle(\widehat{x}, x)$ between $\widehat{x}$ and an exact eigenvector x of $P(\lambda)$ by showing that the sought angle $\angle(\widehat{x}, x)$ is bounded by $\angle(\widehat{y}, y)$, where y is an exact eigenvector of a linearization $L(\lambda) = \lambda X + Y$ and $\widehat{y}$ is its approximation, constructed using $\widehat{x}$ and $\widehat{\lambda}$ to form the Vandermonde structure $\widehat{y} = \Lambda(\widehat{\lambda}) \otimes \widehat{x}$ where $\Lambda(\lambda) = [\phi_{k-1}(\lambda), \phi_{k-2}(\lambda), \ldots, \phi_0(\lambda)]^T$, in which $\{\phi_i(x)\}$ is a polynomial basis specified by the particular linearization $L(\lambda)$: for example, when $L(\lambda)$ is the well-known companion linearization, $\{\phi_i(x)\}$ forms the monomials $\phi_i(x) = x^i$. We can thus obtain an upper bound for $\angle(\widehat{x}, x)$ via that of $\angle(\widehat{y}, y)$, which can be computed via classical perturbation theory for linear eigenvalue problems since $\widehat{y}$ and y are approximate and exact eigenvectors for $L(\lambda)$.

A related problem is to bound the eigenvector perturbation $\angle(x, \widetilde{x})$, where $\widetilde{x}$ is an exact eigenvector of a perturbed matrix polynomial $\widetilde{P}(\lambda) = P(\lambda) + E(\lambda)$, where $E(\lambda)$ is an arbitrary matrix polynomial with "fixed small norm". We show that upper bounds for $\angle(x, \widetilde{x})$ can be obtained by recasting the perturbation $E(\lambda)$ as a backward error in the polynomial eigenvalue problem, then using the above result.

This work shows that just like in linear eigenvalue problems, the eigenvector forward error for polynomial eigenvalue problems is proportional to the residual and inversely proportional to the separation between λ and the rest of eigenvalues of $P(\lambda)$. Another implication is that an eigenvector can be computed accurately even when the same vector corresponds to several distinct eigenvalues.

The results extends to other nonlinear eigenvalue problems such as rational matrix eigenvalue problems, and the bounds can be obtained not only from linearizations but also from other formulations such as quadratization [3, 5].

*University of Tokyo. nakatsukasa@mist.i.u-tokyo.ac.jp
†University of Manchester. francoise.tisseur@manchester.ac.uk

References

[1] M. Berhanu. *The Polynomial Eigenvalue Problem*. PhD thesis, University of Manchester, 2005.

[2] J. P. Dedieu and F. Tisseur. Perturbation theory for homogeneous polynomial eigenvalue problems. *Linear Algebra Appl.*, 358(1):71–94, 2003.

[3] L. Grammont, N. J. Higham, and F. Tisseur. A framework for analyzing nonlinear eigenproblems and parametrized linear systems. *Linear Algebra Appl.*, 435(3):623–640, 2011.

[4] G. W. Stewart and J.-G. Sun. *Matrix Perturbation Theory (Computer Science and Scientific Computing)*. Academic Press, 1990.

[5] F. De Terán, F. M. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the index sum theorem. MIMS EPrint 2013.47, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2013.

[6] F. Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl.*, 309(1):339–361, 2000.

Backward stability of polynomial root-finding using Fiedler companion matrices

Fernando De Terán, Froilán M. Dopico and Javier Pérez

Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree n, with $a_k \in \mathbb{C}$, for $k = 0, \ldots, n-1$. The *first and second Frobenius companion matrices of* $p(z)$ are defined as

$$C_1 := \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_2 := \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{1}$$

and they have the property that $p(z) = \det(zI - C_1) = \det(zI - C_2)$. Hence, the eigenvalues of both C_1 and C_2 coincide with the roots of $p(z)$. In 2003, Fiedler expanded significantly the family of companion matrices associated with a monic polynomial $p(z)$ of degree n (see [3]). These matrices were named *Fiedler matrices* in [1]. The family of Fiedler matrices includes C_1 and C_2 but, provided that $n \geq 3$, the family contains some other different matrices. To construct them, we only need to know the polynomial $p(z)$ and to fix a bijection $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$. We denote the Fiedler matrix associated with the polynomial $p(z)$ and the bijection σ by $M_\sigma(p)$, or M_σ for brevity. These matrices provide a new tool that could be used instead of C_1 and C_2 for computing the roots of $p(z)$.

If the roots of a monic polynomial $p(z)$ are computed as the eigenvalues of a Fiedler matrix M_σ using a backward stable algorithm (for instance, the QR algorithm), the backward stability of the algorithm ensures that the computed roots are the exact eigenvalues of $M_\sigma + E$, where E is a matrix such that

$$\|E\| = O(u)\|M_\sigma\|, \tag{2}$$

for some matrix norm $\|\cdot\|$ and where u denotes the unit roundoff. However, this does not guarantee that these (computed) eigenvalues are the roots of a nearby polynomial of $p(z)$ or, in other words, this does not guarantee that the method is backward stable from the point of view of the polynomials, that is, we would like that the computed eigenvalues should be the exact roots of a polynomial $\widetilde{p}(z)$ such that

$$\frac{\|\widetilde{p} - p\|}{\|p\|} = O(u), \tag{3}$$

for some polynomial norm. In this work, we are interested in the backward stability of the root-finding problem solved via an eigenvalue backward stable method, using Fiedler companion matrices. Our work is motivated by [2], which address related issues for the Frobenius companion matrices.

In order to accomplish this task we need to know how the coefficients of the characteristic polynomial of M_σ change when the matrix is perturbed in the form $M_\sigma + E$. This change can be estimated up to first order in E, through the gradient $\nabla a_k(M_\sigma)$, where $a_k(X) : \mathbb{C}^{n^2} \to \mathbb{C}$ is the kth coefficient of the characteristic polynomial of a matrix $X \in \mathbb{C}^{n \times n}$, considered as a function of its entries. In particular, in this work we compute $\nabla a_k(M_\sigma)$ in terms of the coefficients of $p(z)$. This allows us to get, up to first order, a formula for the variation of the characteristic polynomial of M_σ under small perturbations of M_σ. More precisely, if $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ and $\widetilde{p}(z) = z^n + \sum_{k=0}^{n-1} \widetilde{a}_k z^k$

denotes the characteristic polynomial of M_σ and $M_\sigma + E$, respectively, then we show that, to first order in E,

$$\widetilde{a}_k - a_k = -\sum_{i,j=1}^{n} p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})E_{ij}, \tag{4}$$

where, for $i, j = 1, 2, \ldots, n$, the function $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ is a linear or a quadratic polynomial in the coefficients of $p(z)$. Although the formula for the polynomial $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ is involve in general, we can deduce from (4) several consequences:

(C1) If the roots of a monic polynomial $p(z)$ are computed as the eigenvalues of a Fiedler matrix M_σ using a backward stable eigenvalue algorithm, then, the computed roots are the exact roots of a monic polynomial $\widetilde{p}(z)$ such that

$$\frac{\|\widetilde{p} - p\|_\infty}{\|p\|_\infty} = O(u)\|p\|_\infty^2,$$

if M_σ is a Fiedler matrix other than C_1 or C_2, or

$$\frac{\|\widetilde{p} - p\|_\infty}{\|p\|_\infty} = O(u)\|p\|_\infty,$$

if M_σ is C_1 or C_2.

(C2) When $||p||_\infty = O(1)$, the root-finding problem solved with a backward stable eigenvalue algorithm on any Fiedler companion matrix is a backward stable method.

Note that, when $\|p\|_\infty$ is large, (C1) implies that computing the roots of $p(z)$ using Fiedler matrices is not backward stable. Also, note that (C2) imply that, from the point of view of the backward errors, when $||p||_\infty = O(1)$ Fiedler matrices can be used with the same reliability as the Frobenius companion matrices. Therefore, other Fiedler matrices with particular structures can be used to devise algorithms for computing the roots of $p(z)$. For instance, we could take advantage of the pentadiagonal structure of some Fiedler matrices to devise structured versions of the LR algorithm to get the eigenvalues in $O(n^2)$.

Also, since standard eigenvalue algorithms balance the matrix before computing its eigenvalues, we also consider the process of balancing Fiedler matrices. First, we show how to use (4) to estimate the backward error when the Fiedler matrix is balanced. Second, we provide numerical evidence that show that, if we balance the Fiedler matrix, Equation (4) usually predicts an small backward error, regardless the Fiedler matrix that is used.

Finally, since Fiedler matrices have been defined for monic matrix polynomials in [1], as a next step, we will try to generalize all our results for scalar monic polynomials to matrix monic polynomials.

References

[1] F. De Terán, F. M. Dopico, and D. S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. *SIAM J. Matrix Anal. Appl.*, 31 (2010) 2181–2204.

[2] A. Edelman, H. Murakami. Polynomial roots from companion matrix eigenvalues. *Mathematics of Computation*, 210 (1995) 763–776.

[3] M. Fiedler. A note on companion matrices. *Linear Algebra Appl.,* 372 (2003) 325–331.

Numerical methods for nonlinear two-parameter eigenvalue problems

Bor Plestenjak

We consider the eigenvalue problem

$$\begin{aligned} T_1(\lambda,\mu)x_1 &= 0 \\ T_2(\lambda,\mu)x_2 &= 0, \end{aligned} \tag{1}$$

where $T_1(\lambda,\mu)$ and $T_2(\lambda,\mu)$ are $n_i \times n_i$ complex matrices, whose elements are smooth functions of $\lambda,\mu \in \mathbb{C}$, and $x_1, x_2 \in \mathbb{C}^{n_i}$. Although the matrices T_1 and T_2 could be of different size, we assume that $n_1 = n_2 = n$.

We are searching for nonzero vectors x_1, x_2 and values λ, μ such that (1) is satisfied. In such case the pair (λ,μ) is an eigenvalue and the tensor product $x_1 \otimes x_2$ is the corresponding (right) eigenvector.

The problem (1) is *the nonlinear two-parameter eigenvalue problem (N2EP)*. It can be seen as a generalization of both the nonlinear eigenvalue problem (NEP) and the two-parameter eigenvalue problem (2EP). The eigenvalues of (1) are solutions of the system of bivariate characteristic functions

$$\begin{aligned} \det(T_1(\lambda,\mu)) &= 0 \\ \det(T_2(\lambda,\mu)) &= 0. \end{aligned} \tag{2}$$

As for the NEP, the problem (1) can have zero, finite, or infinite number of eigenvalues. Our only assumption is that the set of eigenvalues is zero-dimensional, i.e., each eigenvalue (λ,μ) is an isolated point.

A motivational example of a N2EP is the quadratic two-parameter eigenvalue problem (Q2EP) [1, 5] of the form

$$\begin{aligned} Q_1(\lambda,\mu)\,x_1 &:= (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda\mu A_{11} + \mu^2 A_{02})\,x_1 = 0 \\ Q_2(\lambda,\mu)\,x_2 &:= (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda\mu B_{11} + \mu^2 B_{02})\,x_2 = 0, \end{aligned} \tag{3}$$

where A_{ij}, B_{ij} are given $n \times n$ complex matrices. A Q2EP of a simpler form, with some of the terms $\lambda^2, \lambda\mu, \mu^2$ missing, appears in the study of linear time-delay systems for the single delay [1].

In the generic case the Q2EP (3) has $4n^2$ eigenvalues that are the roots of the system of the bivariate characteristic polynomials $\det(Q_i(\lambda,\mu)) = 0$ of order $2n$ for $i = 1, 2$. This follows from Bézout's theorem, which states that two projective curves of orders n and m with no common component have precisely nm points of intersection counting multiplicities.

If (λ,μ) is an eigenvalue of N2EP (1), then its geometric multiplicity is

$$m_g((\lambda,\mu)) = \dim(\ker(T_1(\lambda,\mu))) \cdot \dim(\ker(T_2(\lambda,\mu))).$$

Algebraic multiplicity $m_a((\lambda,\mu))$ is the multiplicity of (λ,μ) as a root of the system (2). For an algebraically simple eigenvalue, the Jacobian of (2) has to be nonzero at (λ,μ) and the algebraic simplicity of an eigenvalue implies its geometric simplicity.

If y and x are the left and the right eigenvector of a simple eigenvalue λ_0 of a NEP $A(\lambda)x = 0$, where A is differentiable, then it is well-known that $y^H A'(\lambda)x \neq 0$, see, e.g., [2, 6]. In [4] this relation is

generalized to the N2EP. If (λ_0, μ_0) is an algebraically (and geometrically simple) eigenvalue of the nonlinear two-parameter eigenvalue problem (1), and $x_1 \otimes x_2$ and $y_1 \otimes y_2$ are the corresponding right and left eigenvector, then the matrix

$$\begin{bmatrix} y_1^* \frac{\partial T_1}{\partial \lambda}(\lambda_0, \mu_0) x_1 & y_1^* \frac{\partial T_1}{\partial \mu}(\lambda_0, \mu_0) x_1 \\ y_2^* \frac{\partial T_2}{\partial \lambda}(\lambda_0, \mu_0) x_2 & y_2^* \frac{\partial T_2}{\partial \mu}(\lambda_0, \mu_0) x_2 \end{bmatrix}$$

is nonsingular.

Most of the numerical methods that are available for NEP, can be generalized to N2EP. For an overview of the available numerical methods for NEP, see, e.g., [3, 7]. Here we assume that we have a truly nonlinear 2EP that can not be transformed into a polynomial one. The methods are of course also suitable for polynomial 2EP, which can also be solved by a linearization to a singular 2EP [5]. Among the methods, that can be generalized for N2EP, are:

- Newton's method or the inverse iteration, introduced by Ruhe [7],
- residual inverse iteration by Neumaier [6],
- the method of successive linear problems by Ruhe [7],
- Newton's method for the system (2).

References

[1] E. Jarlebring, M. E. Hochstenbach, *Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations*, Linear Algebra Appl. 431 (2009), pp. 369–380.

[2] P. Lancaster, *Lambda-matrices and vibrating systems*, Pergamon Press, Oxford, 1966.

[3] V. Mehrmann and H. Voss, *Nonlinear eigenvalue problems: A challenge for modern eigenvalue methods*, GAMM-Mitteilungen (GAMM-Reports) 27, 121-152.

[4] A. Muhič and B. Plestenjak, *On the singular two-parameter eigenvalue problem*, Electron. J. Linear Algebra, 18 (2009), pp. 420–437.

[5] A. Muhič and B. Plestenjak, *On the quadratic two-parameter eigenvalue problem and its linearization*, Linear Algebra Appl. 432 (2010), pp. 2529–2542.

[6] A. Neumaier, *Residual inverse iteration for the nonlinear eigenvalue problem.* SIAM J. Numer. Anal., 22 (1985), pp. 914–923.

[7] A. Ruhe, *Algorithms for the nonlinear eigenvalue problem*, SIAM J. Numer. Anal., 10 (1973), pp. 674–689.

Solving matrix polynomials of large degree: Some computational issues

D.A. Bini, University of Pisa
L. Robol, Scuola Normale Superiore, Pisa

February 18, 2014

The solution of certain stochastic models requires the solution of a matrix equation of the kind $X = \sum_{i=0}^{n} A_i X^i$, where X, A_i, for $i = 0, \ldots, n$, are $m \times m$ matrices and n can take very large values [4]. One possible approach for solving this matrix equation relies on computing the roots of the polynomial $\det(xI - \sum_{i=0}^{n} x^i A_i)$ lying in the unit disk in the complex plane. The linearization of this problem leads to a very large block companion matrix whose size is mn.

In this talk we examine some computational aspects related to the numerical computation of the roots of matrix polynomials of large degree with the goal of designing an effective solution algorithm which extends and integrates the package MPSolve. MPSolve is a software for solving polynomial equations, designed more than 14 years ago in the framework of symbolic/numeric computations [1], recently made more effective and fast [2]. It allows to approximate polynomial roots within any guaranteed number of digits even for polynomials of large degrees. The required CPU time is much lower than the time required by the currently available software.

We deal with the following problems:

1. Reducing a block companion matrix in Hessenberg form using the technology of semiseparable matrices [3].
2. Computing the value of the characteristic polynomial $P(\lambda) = \det(\lambda I - H)$ at a point λ and the Newton correction $P(\lambda)/P'(\lambda)$ for a matrix H in Hessenberg form and with a rank structure.
3. Designing a (heuristic) strategy for the choice of initial approximations for the application of Aberth iteration which aims to provide a number of iteration steps independent of the degree.

In particular, concerning 1, we show that any block companion matrix A of block size n with $m \times m$ blocks has a rank-structured Hessenberg form H where the submatrices contained in the upper triangular part have rank at most $k = 2m + 1$. We call k the upper semiseparability rank of H. An algorithm for

computing a semiseparable representation of H is given. It requires $O(m^3n^2)$ ops and uses Gauss transformations. Another algorithm based on the Arnoldi method, is outlined. Finally an algorithm based on Givens rotations for the reduction in Hessenberg form of a block companion matrix in the Chebyshev basis is described. Its cost is still $O(m^3n^2)$ ops.

Concerning 2, we provide backward stable algorithms for computing $P(\lambda) = \det(\lambda I - H)$ and $P(\lambda)/P'(\lambda)$, where H is an $N \times N$ Hessenberg matrix with rank of semiseparability k. Their cost is $O(Nk)$ ops. For the matrices H obtained from block companion matrices where $N = mn$ and $k = 2m + 1$, the cost turns into $O(m^2n)$. This implies that a single step of the Aberth iteration applied to compute the zeros of $P(\lambda)$ costs $O(m^3n^2)$ ops and leads to an effective implementation of the MPSolve algorithm both in the polynomial version and in the "secular" version.

Concerning 3, we provide a heuristic strategy that enables us to choose initial approximations so that the number of steps required in the iterative process is practically independent of nm. The algorithm that we obtain is highly parallelizable. We plan to implement a multithread version in the package MPSolve.

The results of some experiments are shown.

References

[1] D.A. Bini, G. Fiorentino. Design, analysis and implementation of a multiprecision polynomial rootfinder. Numerical Algorithms 23 (2000) 127-173.

[2] D.A. Bini, L. Robol, Solving secular and polynomial equations: A multiprecision algorithm, J. Comput. Appl. Math. Available online 4 May 2013, ISSN 0377-0427, http://dx.doi.org/10.1016/j.cam.2013.04.037.

[3] R. Vandebril, M. Van Barel, and N. Mastronardi, Matrix Computations and Semiseparable Matrices: Linear Systems, The Johns Hopkins University Press , 2007

[4] B. Van Houdt, J.F. Perez. The impact of dampening demand variability in a production/inventory system with multiple retailers, Springer Proceedings in Mathematics & Statistics, Vol. 27, pp. 227–250, 2013.

A Contour Integral-based Parallel Eigensolver for Nonlinear Eigenvalue Problems

Tetsuya Sakurai, Yasunori Futamura, Yasuyuki Maeda, Takahiro Yano and Akira Imakura

In this talk, we present a parallel eigensolver for nonlinear eigenvalue problems that utilize hardware potential of large number of computing nodes.

An eigensolver for generalized eigenvalue problems using contour integral has been proposed in [4]. This method is called the Sakurai-Sugiura (SS) method. This method has a good parallel scalability, and is suitable for massively parallel computing environments.

Recently, we have extended the SS method for polynomial and nonlinear eigenvalue problems [1, 2, 5]. Methods in [1] and [2] derive a generalized eigenvalue problem with a block Hankel structure that includes all the eigenvalues in a given domain on the complex plane. In [5], a Rayleigh-Ritz projection procedure is used to obtain a small size nonlinear eigenvalue problem. The Rayleigh-Ritz type method has a better numerical stability than the Hankel type method, thus we combine these two types of contour integral-based methods to derive a small size generalized eigenvalue problem.

The SS method has several parameters, and the choice of these parameters is crucial for achieving high accuracy and good parallel performance. We also discuss some numerical properties of the method, and present efficient parameter estimation techniques. A stochastic estimation method [3] for eigenvalue count in a domain is used. We demonstrate the efficiency of our method with numerical experiments.

References

[1] J. Asakura, T. Sakurai, H. Tadano, T. Ikegami and K. Kimura, A numerical method for nonlinear eigenvalue problems using contour integrals. *JSIAM Letters*, 1:52–55, 2009.

[2] J. Asakura, T. Sakurai, H. Tadano, T. Ikegami and K. Kimura, A numerical method for polynomial eigenvalue problems using contour integral. *Japan J. Indust. Appl. Math.*, 27:73–90, 2010.

[3] Y. Maeda, Y. Futamura and T. Sakurai, Stochastic estimation method of eigenvalue density for nonlinear eigenvalue problem on the complex plane. *JSIAM Letters*, 3:61–64, 2011.

[4] T. Sakurai and H. Sugiura, A projection method for generalized eigenvalue problems. *J. Comput. Appl. Math.*, 159: 119–128, 2003.

[5] S. Yokota and T. Sakurai, A projection method for nonlinear eigenvalue problems using contour integrals. *JSIAM Letters*, 5:41–44, 2013.

Quadratic eigenvalue problems have too many eigenvalues

Christian Schröder

Projecting a linear eigenvalue problem $A - \lambda B$ using an eigenvector v results in the scalar pencil $v^*Av - \lambda v^*Bv$ whose only eigenvalue is the one corresponding to v. This technique is often used, e.g., to separate the actions of the individual eigenvalues from each other.

Attempting to proceed likewise for a quadratic pencil $\lambda^2 M + \lambda D + K$ with eigenvector v yields the scalar quadratic pencil $\lambda^2 v^*Mv + \lambda v^*Dv + v^*Kv$ which has two eigenvalues: the one corresponding to v, say λ, and an auxiliary one, say μ. This auxiliary Ritz value μ may have nothing to do with eigenvalues of (M, D, K).

The problem was brought to me by a mechanical engineer who wants to solve the second order ODE $M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = 0$ using the ansatz $q(t) = v \cdot \alpha(t)$ for some scalar valued function $\alpha(t)$. The solution is of course $\alpha(t) = \exp(\lambda t)$. However, computing $\alpha(t)$ from the projected ODE can give a second solution $\exp(\mu t)$. This might become a problem if μ dominates λ, i.e., if μ has a larger real part than λ.

Tropical roots as approximations to eigenvalues of matrix polynomials

Meisam Sharify, Françoise Tisseur and Vanni Noferini
(School of Mathematics, University of Manchester, United Kingdom)

Abstract

The tropical roots of $\mathrm{t}_\times p(x) = \max_{0\le j\le \ell} \|A_j\| x^j$ can be good approximations to the moduli of the eigenvalues of the matrix polynomial $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$, in particular when the norm of the matrices A_j have large variations. These tropical roots are the points of non-differentiability of the tropical function $\mathrm{t}_\times p(x)$ and can be computed in only $O(\ell)$ operations. We provide annuli defined in terms of the tropical roots of $\mathrm{t}_\times p(x)$ and containing the eigenvalues of $P(\lambda)$. Our localization results yield conditions under which tropical roots, offer order of magnitude approximations to the moduli of the eigenvalues of $P(\lambda)$. We also show that the tropical roots of $\mathrm{t}_\times p(x)$ are always included inside the bounds of two generalized matrix version of Pellet's theorem. These results also provide further justification for the choice of the starting points in the Ehrlich-Aberth method for the computation of eigenvalues of $P(\lambda)$.

A Backward Stable Algorithm for Quadratic Eigenvalue Problems

Yangfeng Su and Linghui Zeng
School of Mathematical Sciences, Fudan University, Shanghai 200433, China
yfsu@fudan.edu.cn, linghuizeng09@fudan.edu.cn

Quadratic eigenvalue problems (QEPs)

$$(\lambda^2 M + \lambda D + K)x = 0,$$

appear in almost all vibration analysis of systems, such as buildings, circuits, acoustic structures, and so on. We refer the reader to the review paper by Tisseur and Meerbergen[10] in 2001.

QEPs are a subclass of polynomial eigenvalue problems (PEPs). A standard numerical method for the complete solution of a QEP consists of 4 steps,

1. scale the original QEP to another QEP,
2. linearize the scaled QEP as a generalized eigenvalue problem (GEP),
3. solve the GEP by QZ iteration,
4. recover approximated eigenpairs of original QEP from those of the GEP.

Many significant contributions have been made towards a backward stable algorithm for the complete solution of a PEP since 2000. Among them, Tisseur [9] studied the condition number and backward error of PEPs. Mackey, Mackey, Mehl and Mehrmann [8, 7] studied two vector spaces of linearizations of PEPs. Higham, Mackey, Mackey and Tisseur [5] studied symmetric linearizations of PEPs. Higham, Li and Tisseur [4] studied backward error of PEPs solved by linearization.

For QEPs, Fan, Lin and van Dooren [1] proposed a scaling (FLV scaling) which can effectively reduce the condition number of linearized GEP and the backward error in recovery, providing the QEP is not heavily damped, that is $\|D\|/\sqrt{\|M\|\,\|K\|} \leq O(1)$. Gaubert and Sharify [2] proposed another scaling (tropical scaling) which is effective if the QEP is not heavily damped, or the QEP is heavily damped but all system matrices are well conditioned. Higham, Mackey, Tisseur and Garvey [6] vigorously analyzed the sensitivity and backward error in the numerical solution of QEPs, and the effect of scaling. Finally, Hammarling, Munro and Tisseur [3] integrated all these recent advances, and developed some strategies to utilize the special structure in linearization to save computations, i.e., deflating some known zero and infinite eigenvalues to accelerate the convergence and to reduce the backward error of QZ iteration, implemented a MATLAB code `quadeig` for the complete solution of QEPs.

However, for heavily damped QEPs, as stated in [3], examples exist for which after scaling back, the backward errors of some computed eigenvalues are not satisfactory even though some extra strategies were tried.

In this talk, we give an algorithm for solving the complete solution of a QEP.

1. Scale the QEP such that $\|M\| = \|K\| = 1$.
2. Consider two linearizations
$$L_1(\lambda) = \lambda \begin{bmatrix} M & \\ & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & \end{bmatrix}, \quad \widetilde{L}_1(\lambda) = \lambda \begin{bmatrix} M & D \\ & I \end{bmatrix} + \begin{bmatrix} & K \\ -I & \end{bmatrix}.$$
   (a) If the QEP is not heavily damped, choose either one linearization;
   (b) if the QEP is heavily damped, choose L_1 linearization for scaled $|\lambda| \geq 1$ and $\widetilde{L}_1$ linearization for scaled $|\lambda| < 1$.
3. Solve the linearized GEP(s) by QZ iteration.
4. Carefully recover all eigenpairs/eigantriples, and finally scale back.

Guided by above mentioned works, we prove that all computed eigenpairs/eigentriples are backward stable.

References

[1] H.-Y. Fan, W.-W. Lin, and P. Van Dooren. Normwise scaling of second order polynomial matrices. *SIAM J. Matrix Anal. Appl.*, 26(1):252–256, 2004.

[2] S. Gaubert and M. Sharify. Tropical scaling of polynomial matrices. In *Lecture Notes in Control and Inform. Sci.*, volume 389, pages 291–303. Springer, Berlin, 2009.

[3] S. Hammarling, C. J. Munro, and F. Tisseur. An algorithm for the complete solution of quadratic eigenvalue problems. *ACM Trans. Math. Softw.*, 39(3):18:1–18:19, 2013.

[4] N. J. Higham, R.-C. Li, and F. Tisseur. Backward error of polynomial eigenproblems solved by linearization. *SIAM J. Matrix Anal. Appl.*, 29(4):1228–1241, 2007.

[5] N. J. Higham, D. S. Mackey, N. Mackey, and F. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159, 2006.

[6] N. J. Higham, D. S. Mackey, F. Tisseur, and S. D. Garvey. Scaling, sensitivity and stability in the numerical solution of quadratic eigenvalue problems. *Int. J. Numer. Meth. Engng*, 73(3):344–360, 2008.

[7] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Sturctured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[8] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[9] F. Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Alg. Appl.*, 309:339–361, 2000.

[10] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43(2):235–286, 2001.

STRONGLY DAMPED QUADRATIC MATRIX POLYNOMIALS

LEO TASLAMAN

A way to prevent a structure from vibrating rampageously is to incorporate viscous dampers in the design. A viscous damper is a device that produce a damping force proportional to the relative velocity of its ends raised to a power α. We shall consider the case of linear viscous damping, resulting from $\alpha = 1$. This value of α is the default for certain product lines of seismic dampers. The resisting force produced by a viscous damper arises when fluid, trapped in a cylinder, is forced through small holes (see Figure 1). By adjusting the size of these holes, we can make the damper stronger. But stronger is not necessarily better: if a damper is too strong, it resembles a rigid component and hence has little purpose. This means that a structure with only very strong dampers, appears to be quite similar to a structure without dampers. My goal in this talk is to investigate this phenomenon more rigorously for discretized structures. We will do this by studying the eigenvalues and eigenspaces of a related quadratic matrix polynomial.

We shall study the displacements of the nodes in the discretized model, as they vibrate over time. These displacements are given by the solutions to the equations of motion:

$$M\ddot{u}(t) + sD\dot{u}(t) + Ku(t) = 0. \tag{1}$$

Here M, sD and K come from a finite element discretization and correspond to mass, damping and stiffness, respectively. We assume these matrices are real and symmetric positive semi-definite, and further that M and K are strictly

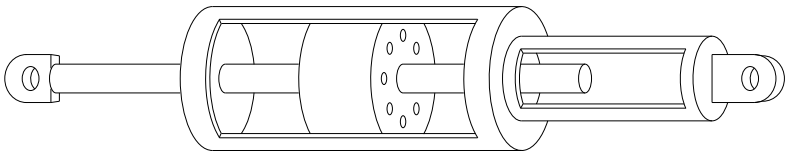


FIGURE 1. A model of a viscous damper. The larger cylinder is filled with a fluid which is forced through holes in the piston head as the piston rod moves horizontally. This causes friction and energy is dissipated and released as heat.

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF MANCHESTER, MANCHESTER, M13 9PL, UK

E-mail address: leo.taslaman@manchester.co.uk.

positive definite. We assume also that the structure we model has r viscous dampers and that each contributes to the damping matrix with a rank one term, so $\operatorname{rank}(D) = r$. If $\|D\| = 1$, say, the parameter s determines the strength of the dampers, so larger s corresponds to dampers with smaller holes, say, and $s = 0$ yields an undamped system. We find the solutions to (1) by solving the quadratic eigenproblem

$$P_s(\lambda)x = 0, \quad s \geq 0, \tag{2}$$

where $P_s(\lambda) = M\lambda^2 + sD\lambda + K$. The spectrum of $P_s(\lambda)$ lies in the left half plane and is symmetric with respect to the real axis. Further, if $(-d + i\omega, x)$ is an eigenpair of $P_s(\lambda)$, where $d, \omega \in \mathbb{R}$, and x is real if $\omega = 0$, then

$$u(t) = e^{-dt}(\cos(t\omega)\operatorname{Re}(x) - \sin(t\omega)\operatorname{Im}(x)) \tag{3}$$

is a real solution to (1) and is called a *mode*. We see that d and ω correspond to damping and frequency, respectively. The solution (3) describes how the model switches between two configurations, given by $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$, as it vibrates. If $x = ve^{i\theta}$ for some $v \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$, then

$$u(t) = e^{-dt}(\cos(t\omega)\cos(\theta) - \sin(t\omega)\sin(\theta))v = e^{-dt}\cos(t\omega + \theta)v,$$

and we see that these two configurations must coincide and that all nodes in the model vibrate in phase. Now, if $s = 0$, it is well known that all eigenvalues are nonzero and purely imaginary, and that all eigenspaces have real bases and are pairwise M-orthogonal. In particular, all modes of an undamped model are undamped and those modes associated with simple eigenvalues are such that all nodes in the model vibrate in phase.

To see the similarities between strongly damped structures and undamped ones, we prove that the eigenvalues of $P_s(\lambda)$ approach nonzero points on the imaginary axis as $s \to \infty$, with the exception of $2r$ real eigenvalues which correspond to strongly overdamped modes. This implies that $n - r$ modes are practically undamped for large enough s. Regarding the eigenspaces of $P_s(\lambda)$ as $s \to \infty$, we show the following. If two eigenvalues converge to distinct points on the imaginary axis, then the corresponding eigenspaces become more and more M-orthogonal in terms of the principal angles. Further, we prove that the span of all eigenvectors associated with eigenvalues converging to a given point has an M-orthonormal basis that become more and more real in the sense that the norms of the imaginary parts go to zero. In particular, eigenvalues converging to points to which no other eigenvalues converge, are, for large enough s, associated with almost real eigenvectors. This corresponds to the case of simple eigenvalues for the undamped problem, and from (3) we see that the associated modes are such that all nodes in the model vibrate essentially in phase.

Solving nonlinear eigenvalue problems in value space

Alex Townsend, University of Oxford

Let $F : [-1,1] \to \mathbb{C}^{n\times n}$ be a Lipschitz continuous function with $\det F(\lambda) \not\equiv 0$ and consider the associated nonlinear eigenvalue problem $F(\lambda)v = 0$, where we seek all the solutions in $[-1,1]$. A powerful and convenient numerical algorithm for solving this problem is to replace $F(\lambda)$ by a matrix polynomial approximant $P_k(\lambda)$ of degree k such that

$$\sup_{\lambda\in[-1,1]} \|F(\lambda) - P_k(\lambda)\|_{\max} \le u \sup_{\lambda\in[-1,1]} \|F(\lambda)\|_{\max}, \tag{1}$$

where u is the unit machine roundoff and $\|\cdot\|_{\max}$ is the absolute maximum entry norm, and then solve the polynomial eigenvalue problem $P_k(\lambda)v = 0$ [4]. Assuming $P_k(\lambda)$ is regular, $P_k(\lambda)v = 0$ can be solved by constructing a generalized eigenvalue problem (GEP), $Av = \lambda Bv$ where $A, B \in \mathbb{C}^{kn\times kn}$, and employing the QZ algorithm.

A matrix polynomial $P_k(\lambda)$ satisfying (1) can be constructed by interpolating $F(\lambda)$ at sufficiently many well-distributed points. More precisely, suppose that $\lambda_0, \ldots, \lambda_k$ are distinct points in $[-1,1]$ such that the unique interpolant of degree k with $P_k(\lambda_j) = F(\lambda_j)$ for $0 \le j \le k$ satisfies (1). We can express $P_k(\lambda)$ by the barycentric formula of the second form,

$$F(\lambda) \approx P_k(\lambda) = \sum_{j=0}^{k} \frac{w_j F_j}{\lambda - \lambda_j} \Big/ \sum_{j=0}^{k} \frac{w_j}{\lambda - \lambda_j}, \qquad F_j = F(\lambda_j) \in \mathbb{C}^{n\times n}, \tag{2}$$

where $\{w_j\}$ are the barycentric weights corresponding to $\{\lambda_j\}$ (for the scalar formula see [2]). From here, provided F_j are nonsingular for $0 \le j \le k$ the following "value space" symmetric arrowhead GEP has the same eigenvalues as $P_k(\lambda)v = 0$ plus $2n$ at infinity (the derivation is omitted):

$$\begin{pmatrix} 0 & F_0 & \cdots & F_k \\ F_0 & \frac{\lambda_0}{w_0}F_0 & & \\ \vdots & & \ddots & \\ F_k & & & \frac{\lambda_k}{w_k}F_n \end{pmatrix} x = \lambda \begin{pmatrix} 0 & & & \\ & -\frac{1}{w_0}F_0 & & \\ & & \ddots & \\ & & & -\frac{1}{w_k}F_k \end{pmatrix} x, \qquad x \in \mathbb{C}^{(k+2)n\times 1}. \tag{3}$$

This can be solved using the QZ algorithm and the eigenvalues outside of $[-1,1]$ discarded to solve the original problem, $F(\lambda)v = 0$.

This symmetric arrowhead GEP may be new, but value space GEPs are not. In 2004 a non-symmetric arrowhead appeared in [3] and very recently two more [1, 8].

Why are value-based GEPs so recent? The companion and colleague matrix have been well-known for several decades and in comparison value-based GEPs are surprisingly recent. A possible explanation is that in the 1960s when such things were being considered for numerical rootfinding, computer hardware threw an error when encountering an infinity and hence, value-based GEPs were not practical. Nowadays, they are just as practical but not as popular.

What interpolation points are allowed? One may wonder if the matrix approximant $P_k(\lambda)$ interpolating $F(\lambda)$ at $\lambda_0, \ldots, \lambda_k$ is guaranteed to satisfy (1) as $k \to \infty$, and the answer depends on both $F(\lambda)$ and $\{\lambda_j\}$. If $F(\lambda)$ is (entrywise) analytic in the so-called "stadium" [7, p. 84], then (1) is always satisfied for sufficiently large k. However, if $F(\lambda)$ is less smooth, such as Lipschitz continuous, then $\lambda_0, \ldots, \lambda_k$ need to be well-distributed and, in particular, interpolation at Chebyshev points converges, but equally-spaced interpolation can diverge.

Acknowledgements

I would like to thank Yuji Nakatsukasa and Vanni Noferini for many discussions related to matrix polynomials. Much of the material presented here is joint work with them.

References

[1] A. Austin, P. Kravanja, and L. N. Trefethen, *Numerical algorithms based on analytic function values at roots of unity*, to appear in SINUM, 2014.

[2] J.-P. Berrut and L. N. Trefethen, *Barycentric Lagrange interpolation*, SIAM Review, 46 (2004), pp. 501–517.

[3] R. Corless, *Generalized companion matrices for the Lagrange basis*, Proceedings EACA, 2004, pp. 317–322.

[4] C. Effenberger and D. Kressner, *Chebyshev interpolation for nonlinear eigenvalue problems*, BIT Numer. Math., 54 (2012), pp. 933–951.

[5] D. S. Mackey, N. Mackey, C. Mehl and V. Mehrmann *Vector spaces of linearizations for matrix polynomials*, SIAM. J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[6] A. Townsend, V. Noferini, Y. Nakatsukasa, *Vector spaces of linearizations for matrix polynomials: a bivariate polynomial approach*, NA Tech. Report, University of Oxford, 2013.

[7] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, 2013.

[8] R. Van Beeumen, W. Michiels, and K. Meerbergen, *Linearization of Lagrange and Hermite interpolating matrix polynomials*, TW Reports, University of Leuven, 2013.

Boundary element methods for resonance problems in open systems

Olaf Steinbach and Gerhard Unger

We analyze the use of boundary element methods for resonance problems in open systems. The characteristic of problems in open systems is that physical quantities of interest, as, e. g., acoustic waves or electric fields, are considered in unbounded domains. Resonance problems in open systems occur for example in the simulation of the vibro-acoustic behavior of ships, in the design of photonic band gap devices for wave guides, or in the modeling of quantum mechanical systems.

As a simple model problem, we consider a time-harmonic acoustic resonance problem formulated as eigenvalue problem of the Dirichlet-Laplacian in an unbounded domain $\Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is bounded a Lipschitz domain:

$$-\Delta p - k^2 p = 0 \quad \text{in } \Omega^c, \qquad p = 0 \quad \text{on } \partial\Omega, \qquad p \text{ "is outgoing"}. \tag{1}$$

For the resonance problem one has to find the eigenpairs $(k, p) \in \mathbb{C} \times H^1_{\rm loc}(\Omega^c) \setminus \{0\}$ of (1). The eigenvalues of (1) are non-real and have a negative imaginary part which describes the damping of the acoustic wave in time. Typically eigenvalues which are close to the real axis are of interest in practical applications.

The basis of boundary element methods is the reduction of boundary value problems which are posed in unbounded domains to their boundaries via boundary integral equations. Such a reduction can be carried out for differential operators where a Green's function can be calculated, e. g., for operators with piecewise constant coefficients. A boundary integral formulation of the resonance problem (1) can be derived from the representation formula for the solution of the Helmholtz equation which reads for problem (1) as follows:

$$p(\tilde{x}) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{e^{ik|\tilde{x}-y|}}{|\tilde{x}-y|} t(y) ds_y, \quad \tilde{x} \in \Omega^c, \tag{2}$$

where $t := \partial_n p$ is the normal derivative of p on the boundary $\partial\Omega$. The application of the trace operator to (2) yields a boundary integral formulation of the eigenvalue problem (1):

$$0 = \frac{1}{4\pi} \int_{\partial\Omega} \frac{e^{ik|x-y|}}{|x-y|} t(y) ds_y =: (V(k)t)(x), \quad x \in \partial\Omega. \tag{3}$$

The eigenvalue problems (1) and (3) are equivalent in the following sense: If (k, p) is an eigenpair of (1), then $(k, \partial_n p)$ is an eigenpair of (3), and vice versa, if $(k, t) \in \mathbb{C} \setminus \mathbb{R} \times H^{-1/2}(\partial\Omega)$ is an eigenpair of (3), then k is an eigenvalue of (1) and the corresponding eigenfunction can be computed by formula (2). The eigenvalue problem (3) is a nonlinear eigenvalue problem with respect to the eigenvalue and we consider it as eigenvalue problem for the operator-valued function $V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))$ [5]. The function V is holomorphic and $V(k)$, $k \in \mathbb{C}$, is a Fredholm operator with index 0. Such kind of eigenvalue problems can be analyzed in the well established framework of eigenvalue problems for holomorphic Fredholm operator-valued functions, see [4] and references therein.

For the approximation of the nonlinear eigenvalue problem (3) a standard Bubnov-Galerkin scheme is applied which requires a discretization of the boundary $\partial\Omega$. In contrast, the application of a finite element based approach to the resonance problem (1) would require a truncation of the unbounded domain Ω^c and the definition of adequate boundary conditions on the artificial boundary which

is critical since resonance functions in the frequency domain are not decaying. The PML method and the Hardy infinite element method are approaches based on the finite element method to the resonance problem (1) and result in generalized eigenvalue problems. However, both approaches give rise to spurious (non-physical) modes which do not occur for the boundary element approach [5].

Comprehensive convergence results of the boundary element method for the nonlinear eigenvalue problem (3) including error estimates of the eigenvalue and eigenfunctions are presented in [5]. These results are applied in [3] to a coupled finite and boundary element approach of a fluid-solid interaction eigenvalue problem. Numerical experiments in [6] indicates that the boundary element method is also a promising approach for Maxwell's eigenvalue problem.

The boundary element method for the eigenvalue problem (3) results in a holomorphic matrix eigenvalue problem of the form

$$V_h(k_h)\underline{t} = \underline{0}, \tag{4}$$

with the matrix

$$V_h(k_h)[i,j] = \frac{1}{4\pi}\int_{\partial\Omega}\int_{\partial\Omega}\frac{e^{ik|x-y|}}{|x-y|}\varphi_j(y)ds_y\varphi_i(x)ds_x,$$

where $\varphi_1,\ldots,\varphi_n$ is a basis of the approximation space for the eigenfunctions, e. g., the space of piecewise polynomial functions on $\partial\Omega$. For small and medium sized problems the contour integral method and an interpolation based approach have been applied to the eigenvalue problem (4) in [5, 1] and to a coupled FEM-BEM eigenvalue problem arising form a fluid-solid interaction problem in [3, 2]. Since boundary element matrices are dense and the computation of the entries is expensive, techniques which allow an efficient compression of matrices are mandatory for large sized problems. Such techniques are the fast multipole method, the adaptive cross approximation or hierarchical matrices. An exact factorization of boundary element matrices by using these techniques is not possible. Therefore the practical solution of large sized nonlinear eigenvalue problems arising from boundary element discretizations of PDEs is still a demanding challenge.

References

[1] C. Effenberger, D. Kressner. Chebyshev interpolation for nonlinear eigenvalue problems. *BIT*, 52:933–951, 2012.

[2] C. Effenberger, D. Kressner, O. Steinbach, and G. Unger. Interpolation-based solution of a nonlinear eigenvalue problem in fluid-structure interaction. *Proceedings in Applied Mathematics and Mechanics*, 12:633–634, 2012.

[3] A. Kimeswenger, O. Steinbach, G. Unger. Coupled finite and boundary element methods for fluid-solid interaction eigenvalue problems. *Preprints of the Institute of Computational Mathematics of TU Graz*, No. 9, 2013.

[4] V. Kozlov, V. Maz′ya: Differential equations with operator coefficients with applications to boundary value problems for partial differential equations. Springer, Berlin, 1999.

[5] O. Steinbach, G. Unger. Convergence analysis of a Galerkin boundary element method for the Dirichlet Laplacian eigenvalue problem. *SIAM J. Numer. Anal.*, 80:710–728, 2012.

[6] Ch. Wieners, J. Xin. Boundary element approximation for Maxwell's eigenvalue problem. *Math. Meth. Appl. Sci.*, 36:2524–2539, 2013.

Compact Rational Krylov methods for solving nonlinear eigenvalue problems

Roel Van Beeumen, Karl Meerbergen and Wim Michiels

We present a new framework of Compact Rational Krylov (CORK) methods for solving the nonlinear eigenvalue problem (NLEP):

$$A(\lambda)x = 0.$$

For many years linearizations are used for solving polynomial eigenvalue problems [4]. On the other hand, for the general nonlinear case, $A(\lambda)$ can first be approximated by a (rational) matrix polynomial and then a convenient linearization is used [1, 3, 6, 2]. The major disadvantage of methods based on linearizations is the growing memory cost with the iteration count, i.e., in general the memory cost is proportional to the degree of the polynomial. However, the CORK family of rational Krylov methods exploits the structure of the linearization pencils and uses a generalization of the compact Arnoldi decomposition [5]. In this way, the extra memory cost due to the linearization of the original eigenvalue problems is negligible.

References

[1] C. Effenberger and D. Kressner, *Chebyshev interpolation for nonlinear eigenvalue problems*, BIT, 52 (2012), pp. 933–951.

[2] S. Güttel, R. Van Beeumen, K. Meerbergen, and W. Michiels, *NLEIGS: A class of robust fully rational Krylov methods for nonlinear eigenvalue problems*, tech. report TW633, Department of Computer Science, KU Leuven, Leuven, 2013.

[3] E. Jarlebring, W. Michiels, and K. Meerbergen, *A linear eigenvalue algorithm for the nonlinear eigenvalue problem*, Numer. Math., 122 (2012), pp. 169–195.

[4] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Vector Spaces of Linearizations for Matrix Polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[5] Y. Su, J. Zhang, and Z. Bai, *A compact Arnoldi algorithm for polynomial eigenvalue problems*, in Recent Advances in Numerical Methods for Eigenvalue Problems (RANMEP2008), Taiwan, Jan. 2008.

[6] R. Van Beeumen, K. Meerbergen, and W. Michiels, *A Rational Krylov Method Based on Hermite Interpolation for Nonlinear Eigenvalue Problems*, SIAM J. Sci. Comput., 35 (2013), pp. A327–A350.

Variational characterization of eigenvalues of a non–symmetric eigenvalue problem governing elastoacoustic vibrations

Markus Stammberger, Heinrich Voss

In this talk we consider the elastoacoustic vibration problem, which consists of determining the small amplitude vibration modes of an elastic structure coupled with an internal inviscid, homogeneous, compressible fluid.

Different formulations have been proposed to model this problem, the most obvious of which describes the structure by its relative displacement field u and the fluid by its pressure p. Thus one arrives at the following system of homogeneous time-independent partial differential equations

$$\begin{aligned} \text{Div } \sigma(u) + \omega^2 \rho_s u &= 0 \text{ in } \Omega_s, \\ \nabla^2 p + \frac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f, \\ u &= 0 \text{ on } \Gamma_D, \\ \nabla p \cdot n_f &= 0 \text{ on } \Gamma_N, \\ \sigma(u)\, n - p\, n &= 0 \text{ on } \Gamma_I, \\ \omega^2 \rho_f u \cdot n + \nabla p \cdot n &= 0 \text{ on } \Gamma_I, \end{aligned} \tag{1}$$

where Ω_s and Ω_f denotes the region occupied by the structure and the fluid, respectively. Γ_D and Γ_N are Dirichlet– and Neumann-parts of the outer boundary of the structure, and Γ_I the interface between the fluid and the structure. The interface boundary conditions are a consequence of an equilibrium of acceleration and force densities at the contact interface.

Although this eigenvalue problem is not self–adjoint it shares many important properties with self–adjoint models: It has a countable set of eigenvalues which are real and non–negative, and taking advantage of a Rayleigh functional (which generalizes the Rayleigh quotient for self–adjoint problems) its eigenvalues allow for the variational characterizations known from the symmetric theory [3]. Namely, they can be characterized by Rayleigh's principle, and are minmax and maxmin values of the Rayleigh functional.

Discretizing the elastoacoustic problem with finite elements where the triangulation obeys the geometric partition into the fluid and the structure domain one obtains a non–symmetric matrix eigenvalue problem which inherits the variational properties.

$$Kx := \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} =: \lambda M x. \tag{2}$$

The following properties can be proved:

- The eigenvalues of the discrete problem (2) are upper bounds of the corresponding eigenvalues of problem (1).
- The standard spectral approximation theory applies to prove convergence results for Galerkin type methods.
- Eigenfunctions (of problem (1) and of its adjoint problem) can be chosen to satisfy an orthogonality property.

- A Krylov–Bogoliubov type eigenvalue bound holds [4].
- For the matrix eigenvalue problem the Rayleigh functional iteration is cubically convergent as is the Rayleigh quotient iteration for linear symmetric problems [1].
- Based on the variational characterization structure preserving iterative projection methods of Jacobi–Davidson type and nonlinear Arnoldi type can be defined [1, 4].
- The automated multi-level sub-structuring method (AMLS) introduced by Bennighof for linear symmetric eigenvalue problems in structural analysis can be generalized to the non–symmetric elastoacoustic problem, and an a priori error bound can be proved using the minmax characterization [2].

References

[1] M. Stammberger, H. Voss. On an unsymmetric eigenvalue problem governing free vibrations of fluid-solid structures. *Electr. Trans. Numer. Anal.*, 36:113–125, 2010.

[2] M. Stammberger, H. Voss. Automated Multi-Level Sub-structuring for fluid-solid interaction problems. *Numer. Lin. Alg. Appl.*, 18:411–427, 2011.

[3] M. Stammberger, H. Voss. Variational characterization of eigenvalues of a non–symmetric eigenvalue problem governing elastoacoustic vibrations. *Appl. Math.*, 59:1–13, 2014.

[4] H. Voss, M. Stammberger. Structural-acoustic vibration problems in the presence of strong coupling. *J. Pressure Vessel Techn.*, 135:011303 (8 pages), 2013.

Structure Preserving Transformation for Quadratic Matrix Polynomials

Ion Zaballa

A well-known result in the spectral theory of matrix polynomials states that two $m \times n$ matrix polynomials are equivalent if and only if they have the same rank and the same elementary divisors. This is the first of several crucial results that culminate in the computation of the eigenvalues of real or complex square matrices. Specifically,

(L1) If A_1 and A_2 are nonsingular then $\lambda A_1 + B_1$ and $\lambda A_2 + B_2$ are equivalent if and only if there are nonsingular matrices P and Q such that $(\lambda A_1 + B_1)P = Q(\lambda A_2 + B_2)$. In particular, $\lambda I - B_1$ and $\lambda I - B_2$ are equivalent if and only if B_1 and B_2 are similar; i.e., there is a nonsingular P such that $B_1 P = P B_2$.

(L2) "Most" matrices are similar to diagonal matrices and so, the diagonal elements of the latter ones convey the information about the elementary divisors (in particular, the eigenvalues) of the original matrices. In general, however, similarity is not suitable for the numerical computation of the eigenvalues.

(L3) All complex matrices are unitarily similar to triangular matrices revealing their eigenvalues. Unitary similarity is suitable, in general, for eigenvalue computation.

Do we have similar results for quadratic matrix polynomials? At some extent the answer is in the affirmative. Justifying this assertion is the goal of the following lines.

Let us see first how (L1) generalizes to quadratic matrix polynomials. Assume that $Q(\lambda) = Q_2\lambda^2 + Q_1\lambda + Q_0, \widehat{Q}(\lambda) = \widehat{Q}_2\lambda^2 + \widehat{Q}_1\lambda + \widehat{Q}_0 \in \mathbb{F}[\lambda]^{n\times n}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and Q_2, $\widehat{Q}_2$ nonsingular matrices. Then one can prove (see Theorem 5.1 of [1]) that $Q(\lambda)$ and $\widehat{Q}(\lambda)$ are equivalent matrix polynomials if and only if there are matrix binomials $F(\lambda) = F_1\lambda + F_2$ and $\widehat{F}(\lambda) = \widehat{F}_1\lambda + \widehat{F}_2$ such that the following two conditions hold:

(i) $Q(\lambda)\widehat{F}(\lambda) = F(\lambda)\widehat{Q}(\lambda)$.

(ii) $(Q(\lambda), F(\lambda))$ are left coprime and $(\widehat{Q}(\lambda), \widehat{F}(\lambda))$ are right coprime.

Thus, $F(\lambda)$ and $\widehat{F}(\lambda)$ play the role of P and Q in (L1). It turns out that the existence of binomials $F(\lambda)$ and $\widehat{F}(\lambda)$ satisfying (i) and (ii) is equivalent to the existence of Structure Preserving Transformations for some linearizations of $Q(\lambda)$ and $\widehat{Q}(\lambda)$. We will use the *left companion linearization* to show this equivalence but some other linearizations can be used as well (see [1, Sec. 3]). Also, in order to simplify the notation, we will assume from now on that $Q_2 = \widehat{Q}_2 = I_n$ and so $F_1 = \widehat{F}_1$. Let $A = \begin{bmatrix} 0 & -Q_0 \\ I_n & -Q_1 \end{bmatrix}$ and $\widehat{A} = \begin{bmatrix} 0 & -\widehat{Q}_0 \\ I_n & -\widehat{Q}_1 \end{bmatrix}$ be the left companion matrices of $Q(\lambda)$ and $\widehat{Q}(\lambda)$, respectively. A Structure Preserving Transformation (SPT) form A to $\widehat{A}$ is any nonsingular matrix S such that $AS = S\widehat{A}$. It is easily seen that the SPTs, S, from A to $\widehat{A}$ are characterized by the following two properties (provided that S is nonsingular):

$$\text{(I)} \quad S = \begin{bmatrix} Y & AY \end{bmatrix}, \quad \text{(II)} \quad A^2Y + AY\widehat{Q}_1 + Y\widehat{Q}_0 = 0.$$

In other words, (Y, A) is a *left standard pair* of $\widehat{Q}(\lambda)$ (see [2]). What is not so obvious is that if $S = \begin{bmatrix} F_0 & X_0 \\ F_1 & X_1 \end{bmatrix}$ is an SPT from A to $\widehat{A}$ then $F(\lambda) = F_1\lambda + F_0$ and $\widehat{F}(\lambda) = F_1\lambda + (F_0 + F_1\widehat{Q}_1 - Q_1F_1)$

are binomials that satisfy (i) and (ii). And conversely, if $F(\lambda) = F_1\lambda + F_0$ and $\widehat{F}(\lambda) = F_1\lambda + \widehat{F}_0$ are binomials that satisfy (i) and (ii) then $\widehat{F}_0 = F_0 + F_1\widehat{Q}_1 - Q_1F_1$ and there are $n \times n$ matrices X_0, X_1 such that $S = \begin{bmatrix} F_0 & X_0 \\ F_1 & X_1 \end{bmatrix}$ is an SPT from A to $\widehat{A}$ (see [1]) . Thus, there is a very tight relationship between similarity transformations preserving left companion matrices and binomials preserving the equivalence of quadratic matrix polynomials.

As far as a generalization of (L2), "most" quadratic matrix polynomials (complex or real) are *equivalent* to quadratic diagonal matrices (see [3]). Hence, there are binomials satisfying (i) and (ii) with $\widehat{Q}(\lambda) = D(\lambda)$ a quadratic diagonal matrices. And there are SPTs from A to $\widehat{A}$ with $\widehat{Q_0}$ and $\widehat{Q}_1$ diagonal matrices. Since similarity, in general, is not suitable for the numerical computation of eigenvalues of square matrices, it is hopeless, in general, to try the use of general SPTs to obtain the eigenvalues of A.

We are going to see next that the generalization of (L3) for quadratic matrix polynomials indicates that there are some types of SPTs that might be worth-investigating regarding the computation of the eigenvalues of quadratic matrix polynomials. In fact, it was shown in [4] that **any** complex quadratic matrix polynomials and **almost any** real quadratic matrix polynomials, $Q(\lambda)$, is equivalent to a triangular quadratic matrix polynomials, $T(\lambda)$. In addition, if A and T are the left companion matrices of $Q(\lambda)$ and $T(\lambda)$, then there are SPTs $S = \begin{bmatrix} U & AU \end{bmatrix}$ from A to T, where U is a matrix whose columns form an orthonormal system of vectors. Let us call Orthogonal Structure Preserving Transformations (OSPT) this special type of SPTs. Some questions arise in a natural way: Can OSPTs be used to produce a sequence of similar left companion matrices A_k ($A_0 := A$) converging to a left companion matrix T of a triangular quadratic matrix polynomial? Are the OSPTs suitable for the numerical computation of a triangular form of *some* quadratic matrix polynomials? The following result is a first step in providing an affirmative answer to the first question.

Theorem 1 *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a nonderogatory quadratic matrix polynomial and let A be its left companion matrix. There exists an OSPT $S = \begin{bmatrix} u_1 & \cdots & u_n & Au_1 & \cdots Au_n \end{bmatrix}$ such that if $S_j = \begin{bmatrix} u_1 & \cdots u_j & Au_1 & \cdots Au_j \end{bmatrix}$, then*

$$AS_j = S_jT_j + t_{j+1\,2j}u_{j+1}e_{2j}^T, \quad j = 1, \ldots n,$$

where $T_j = \begin{bmatrix} 0 & -H_j \\ I_j & -R_j \end{bmatrix}$, $H_j \in \mathbb{F}^{j\times j}$ is an upper Hessenberg matrix, $R_j \in \mathbb{F}^{j\times j}$ is an upper triangular matrix, $e_{2j}^T = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{F}^{1\times 2j}$ and $u_{n+1} = 0$.

Som implications and practical implementations of this result will be analyzed in the talk.

References

[1] S. D. Garvey, P. Lancaster, A. A. Popov, U. Prells, and I. Zaballa, Filters connecting isospectral quadratic systems. *Linear Alg. Appl.*, 438: 1497–1516, 2013.

[2] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*. Academic Press, New York, 1982 and SIAM, Philadelphia, 2009.

[3] P. Lancaster, and I. Zaballa, Diagonalizable quadratic eigenvalue problems. *Mechanical Systems and Signal Processing*, 23: 1134–1144, 2009.

[4] F. Tisseur, I. Zaballa, Triangularizing quadratic matrix polynomials, *SIAM J. Matrix Anal. Appl.*, 34(2): 312–337, 2013.