Symmetries and classification of integrable nonlinear PDEs

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How do we test whether a given system is integrable and if so, how can we integrate it?

What are the integrability conditions?

Can we describe all integrable systems of a certain type (classification problem)?

Can we give a complete picture of all possible integrable systems of all orders (global classification)?

To answer these challenging questions we ought to decide what integrability is.

To classify equations we have to define the equivalence relation, and ideally give a method to check whether two given equations are equivalent or not.

Symmetry approach to the problem of Integrability

- Elements of differential algebra and jet calculus.

- Symmetries and local conservation laws

- Formal series, Adler's Theorem

- A concept of formal recursion operator and formal simplectic operator.

- Integrability conditions.

- The problem of classification of Integrable Equations.

- Some results and generalizatios.

PDEs as dynamical systems. Dynamical variables.

Differential field: Denote by $\mathfrak{F} = \mathfrak{F}(t, x, u; D)$ the field of "all" smooth complex functions of a finite number of *independent* jet variables

$$t\,, x,\, u_0 = u,\, u_1 = u_x,\, u_2 = u_{xx}, \ldots, \tag{1}$$

equipped with the derivation $D : \mathfrak{F} \mapsto \mathfrak{F}$

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \cdots . \tag{2}$$

which represents the total derivative with respect to x (the chain rule). We assume that $\mathbb{C} \subset \mathfrak{F}$.

Elements of $\mathfrak{F}$ we call **local** functions.

In this notations a scalar evolutionary equation

$$u_t = F(t, x, u, u_x, \dots, \partial_x^n u)\,. \tag{3}$$

for a dependent variable $u = u(x,t)$ and independent variables t, x takes the form

$$u_t = F(t, x, u, u_1, \dots, u_n)\,, \qquad F \in \mathfrak{F}\,. \tag{4}$$

Evolution partial differential equation (4) defines another derivation D_t of the field $\mathfrak{F}$

$$D_t = \frac{\partial}{\partial t} + F_0 \frac{\partial}{\partial u_0} + F_1 \frac{\partial}{\partial u_1} + F_2 \frac{\partial}{\partial u_2} + \cdots\,, \qquad F_k \in \mathfrak{F}$$

where

$$F_0 = F(t, x, u, ..., u_n), \qquad F_1 = D(F_0), \qquad ... \qquad F_n = D^n(F_0), \qquad ... \,.$$

The vector field D_t represents the total derivative with respect to time t due to evolutionary equation (4). Derivations D_t and D commute:

$$[D_t, D] = \sum_{s=0} \Big(D_t(u_{s+1}) - D(F_s)\Big) \frac{\partial}{\partial u_s} = \sum_{s=0} \Big(F_{s+1} - D(F_s)\Big) \frac{\partial}{\partial u_s} = 0\,.$$

Evolutionary equation (4) can be represented by two compatible infinite dimensional dynamical systems

$$D(u_s) = u_{s+1}\,, \qquad D_t(u_s) = F_s\,, \qquad s = 0, 1, 2, \ldots\,. \tag{5}$$

Variables $u, u_1, u_2, \ldots$ we call *dynamical variables*.

Example: For the Korteweg de Vries equation

$$u_t = u_{xxx} + 6uu_x \tag{6}$$

function $F_0 = u_3 + 6uu_1$ and first few equations of the system (5) are of the form

$$\begin{aligned}
&D(u) = u_1\,, \quad D_t(u) = u_3 + 6uu_1 \\
&D(u_1) = u_2\,, \quad D_t(u_1) = u_4 + 6uu_2 + 6u_1^2 \\
&D(u_2) = u_3\,, \quad D_t(u_2) = u_5 + 6uu_3 + 18u_1u_2 \\
&\ldots
\end{aligned}$$

We say that a derivations of the field $\mathfrak{F}$

$$D_G = G_0 \frac{\partial}{\partial u_0} + G_1 \frac{\partial}{\partial u_1} + G_2 \frac{\partial}{\partial u_2} + \cdots , \qquad G_k \in \mathfrak{F}$$

is an evolutionary derivation, if $[D, D_G] = 0$. It follows from the condition $[D, D_G] = 0$ that $G_k = D^k(G_0)$.

With any evolutionary equation (where τ is a new independent variable)

$$u_\tau = G(t, x, u, u_1, \ldots, u_m), \qquad G \in \mathfrak{F} \tag{7}$$

we associate the corresponding evolutionary derivaion

$$D_G = G \frac{\partial}{\partial u_0} + D(G) \frac{\partial}{\partial u_1} + D^2(G) \frac{\partial}{\partial u_2} + \cdots .$$

We note that $D = \frac{\partial}{\partial x} + D_{u_1}$, $D_t = \frac{\partial}{\partial t} + D_F$.

All evolutionary derivations form a Lie algebra with respect to the standard commutator

$$D_K = D_G \circ D_H - D_H \circ D_G = [D_G, D_H] \tag{8}$$

of vector fields.

The generator K of the commutator is given by

$$K = D_G(H) - D_H(G). \tag{9}$$

It defines a Lie bracket $[G, H] = D_G(H) - D_H(G)$ on the differential field $\mathfrak{F}$. The field $\mathfrak{F}$ equiped with this Lie bracket is an infinite dimensional Lie algebra.

Fréchet derivative and Euler's operator

Definition 1. *For any element $a \in \mathfrak{F}$ the Fréchet derivative is defined as a linear differential operator of the form*

$$a_* = \sum_k \frac{\partial a}{\partial u_k} D^k \,.$$

The order of function a is defined as the order of the differential operator a_* (i.e. the maximal power of D).

Using this notation the Lie bracket $[G, H]$ on $\mathfrak{F}$ can be written as

$$[G, H] = H_*(G) - G_*(H).$$

We denote a_*^+ the formally conjugated operator

$$a_*^+ = \sum_k (-1)^k D^k \circ \frac{\partial a}{\partial u_k}\,.$$

Definition 2. *The Euler operator or the variational derivative of* $a \in \mathfrak{F}$ *is defined as*

$$\frac{\delta a}{\delta u} = \sum_k (-1)^k D^k \left(\frac{\partial a}{\partial u_k}\right) = a_*^+(1)\,.$$

We say that $a \in \mathfrak{F}$ is a total derivative if $a \in \mathrm{Im}(D)$, and $\mathrm{Im}(D)$ is defined as the image $D : \mathfrak{F} \to \mathrm{Im}(D)$ of the derivation D.

If function a is a total derivative $a = D(b), b \in \mathfrak{F}$ then the variational derivative vanishes. Moreover the vanishing of the variational derivative is almost a criteria that the function belongs to $\mathrm{Im}(D)$.

Theorem 1. *[Gelfand, Manin, Shubin] For* $a \in \mathfrak{F}(u; D)$ *the variational derivative vanishes*

$$\frac{\delta a}{\delta u} = 0$$

if and only if $a \in$ *Im*$(D) + \mathbb{C}$.

Here we list some **useful identities:**

For any $a, b, F \in \mathfrak{F}$.

$$D_b(a) = a_*(b)\,, \tag{10}$$

$$D(a) = \partial_x a + a_*(u_1)\,, \tag{11}$$

$$D_t(a) = \partial_t a + a_*(F)\,, \tag{12}$$

$$(ab)_* = ab_* + ba_*\,, \tag{13}$$

$$(D(a))_* = D \circ a_* = D(a_*) + a_* \circ D\,, \tag{14}$$

$$(D_t(a))_* = D_t(a_*) + a_* \circ F_*\,, \tag{15}$$

$$(a_*(b))_* = D_b(a_*) + a_* \circ b_*\,, \tag{16}$$

$$\left(\frac{\delta a}{\delta u}\right)_* = \left(\frac{\delta a}{\delta u}\right)_*^+, \tag{17}$$

$$\frac{\delta}{\delta u}(D_t(a)) = D_t\left(\frac{\delta a}{\delta u}\right) + F_*^+\left(\frac{\delta a}{\delta u}\right). \tag{18}$$

Here $A \circ B$ denotes a composition of operators A and B.

Local symmetries and conservation laws.

Traditionally symmetries of equations are defined as transformations which map solutions of the equation into solutions.

Suppose u is an arbitrary solution of equation

$$u_t = F(t, x, u, u_1, \ldots, u_n)\,, \qquad F \in \mathfrak{F}\,. \tag{19}$$

Let us consider an infinitesimal transformation

$$\hat{u} = u + \tau G(t, x, u, \ldots, u_m)\,, \qquad G \in \mathfrak{F}\,, \tag{20}$$

which depends on a small parameter τ. We say that the transformation (20) defines an infinitesimal symmetry of equation (19) and G is a generator of the symmetry, if $\hat{u}$ satisfies equation

$$\hat{u}_t = F(t, x, \hat{u}, \ldots, \hat{u}_n) + O(\tau^2)\,.$$

If we substitute $\hat{u}$ (20) in (19) and request the cancellation of terms of order τ, we get

$$D_t(G(t, x, u, \ldots, u_m)) = F_*(G(t, x, u, \ldots, u_m))\,. \tag{21}$$

Equivalent definitions of symmetry:

We say that $G \in \mathfrak{F}$ is a generator of an infinitesimal symmetry (or simply G is a symmetry) for equation $u_t = F$, if

1. For any solution u of the equation $u_t = F(t, x, u, u_1, \ldots)$, function $\hat{u} = u + \tau G$ satisfies to $\hat{u}_t = F(t, x, \hat{u}, \hat{u}_1, \ldots) + O(\tau^2)$. $\Leftrightarrow$

2. $D_t(G) = F_*(G)$. $\Leftrightarrow$

3. $\frac{\partial G}{\partial t} + G_*(F) - F_*(G) = 0$. $\Leftrightarrow$

4. Two evolutionary equations $u_t = F$ and $u_\tau = G$ are compartible. $\Leftrightarrow$

5. $[D_t, D_G] = 0$.

In literature the equation $u_\tau = G$, which is compartible with $u_t = F$, is often called a symmetry of $u_t = F$.

Suppose we have two symmetries D_G and D_H, then the commutator $[D_G, D_H]$ also commutes with D_t and D due to the Jacobi identity for vector fields and therefore is a symmetry. A linear combination of symmetries with constant coefficients is a symmetry. In other words, infinitesimal symmetries of an equation form a Lie algebra over $\mathbb{C}$, which is a subalgebra of the Lie algebra of all evolutionary derivations.

Definition 3. *Let G be a symmetry. The order of the differential orperator G_* is called the order of the symmetry G.*

The new time τ in $u_\tau = G$ plays role of a group parameter. In order to find an one-parameter family $u(x,t,\tau)$ of solutions including a given solution $u(x,t)$ of the equation $u_t = F$, we have to solve the equation $u_\tau = G$ with the initial data $u(x,t,0) = u(x,t)$.

Example: For the KdV equation

$$u_t = u_3 + 6uu_1$$

we have obvious symmetries with generators

$$G_1 = u_1, \qquad G_3 = u_t = u_3 + 6uu_1$$

of order 1 and 3 which correspond to the shifts in space and time. Indeed,the corresponding evolutionary equations

$$u_{\tau_1} = u_1, \qquad u_{\tau_3} = u_t = u_3 + 6uu_1$$

can be easily integrated to a group action

$$\widehat{u}(x,t,\tau_1) = u(x+\tau_1,t), \qquad \widehat{u}(x,t,\tau_3) = u(x,t+\tau_3).$$

The Galileian and scaling transformations are generated by

$$G_g = 1 + 6tu_1\,, \quad G_s = 2u + xu_1 + 3t(u_3 + 6uu_1).$$

Integration of the equation $u_{\tau_g} = G_g$ leads to

$$\widehat{u}(x,t,\tau_g) = u(x + 6\tau_g t, t) + \tau_g.$$

There are infinitely many high order symmetries for the KdV equation, which cannot be integrated to a group action explicitely. The first nontrivial one has order 5 and is of the form

$$G_5 = u_5 + 10uu_3 + 20u_1u_2 + 30u^2u_1\,.$$

Example: For any m equation $u_\tau = u_m$ is a symmetry for $u_t = u_n$.

Example: The Burgers equation

$$u_t = u_{xx} + 2uu_x$$

has the following third order symmetry

$$u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x.$$

It is easy to verify that all these functions are indeed generators of symmetries according the definitions given above.

(* symmetry reductions *)

Conservation laws.

The notion of first integrals, in contrast to symmetries, cannot be generalized to the case of PDEs. It is replaced by the concept of local conservation laws, which are also related to constants of motion.

Definition. A function $\rho \in \mathfrak{F}$ is called a *density* of a local conservation law of an evolutionary equation if there exist a function $\sigma \in \mathfrak{F}$ such that

$$D_t(\rho) = D(\sigma). \tag{22}$$

The function σ is called a *flux* of the conservation law.

Example. Functions

$$\rho_1 = u, \quad \rho_2 = u^2, \quad \rho_3 = -u_1^2 + 2u^3$$

are conserved densities of the Korteweg - de Vries equation

$$u_t = u_3 + 6uu_1.$$

Indeed,

$$D_t(u) = D(u_2 + 3u^2),$$

$$D_t(u^2) = D(2uu_2 - u_1^2 + 4u^3),$$

$$D_t(\rho_3) = D(9u^4 + 6u^2u_2 + u_2^2 - 12uu_1^2 - 2u_1u_3)\,.$$

Function u^3 is not a density of a conservation law for the Korteweg de Vries equation. Indeed, $D_t(u^3) = 3u^2u_3 + 18u^3u_1$. In order to check that the right-hand side is not a total derivative we apply the Euler operator

$$\frac{\delta}{\delta u}(3u^2u_3 + 18u^3u_1) = -18u_1u_2 \neq 0\,.$$

If u is a function periodic in space variable x with period L, then $I_k = \int_0^L \rho_k\, dx$ do not depend on time and are constants of motion.

Relation $D_t(\rho) = D(\sigma)$ is evidently satisfied if $\rho = D(h)$ for any $h \in \mathfrak{F}$. In this case $\sigma = D_t(h)$. Such "conservation laws" we call trivial.

Definition. Two conserved densities ρ_1, ρ_2 are called equivalent $\rho_1 \sim \rho_2$ if the difference $\rho_1 - \rho_2$ is a trivial density (i.e. $\rho_1 - \rho_2 \in \mathrm{Im} D$).

Definition. The *order* ord(ρ) of a conserved density ρ is defined as the order of the differential operator

$$R = \left(\frac{\delta\rho}{\delta u}\right)_* .$$

For trivial densities $\delta\rho/\delta u = 0$ and therefore equivalent densities have the same order. For example, densities $\rho_1 = u_1^2 + u_3$ and $\rho_2 = -uu_2$ are equivalent and ord$(\rho_1) =$ ord$(\rho_2) = 2$.

A linear combination of conserved densities with constant coefficients is also a conserved density. Therefore the set of conserved densities form a linear space, actually a factor space over $\mathrm{Im} D$.

Formal pseudo-differential series.

For further consideration we will need formal pseudo-differential series, which for simplicity we shall call formal series (of order $m = \mathrm{ord} A$)

$$A = a_m D^m + a_{m-1} D^{m-1} + \cdots + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \cdots \qquad a_k \in \mathfrak{F}\,. \tag{23}$$

The product of two formal series is defined by

$$aD^k \circ bD^m = a(bD^{m+k} + C_k^1 D(b) D^{k+m-1} + C_k^2 D^2(b) D^{k+m-2} + \cdots)\,, \tag{24}$$

where $k, m \in \mathbb{Z}$ and C_n^j is the binomial coefficient

$$C_n^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}\,.$$

This product is associative.

The formally conjugated to (23) formal series A^+ is defined as

$$A^+ = (-1)^m D^m \circ a_m + \cdots + a_0 - D^{-1} \circ a_{-1} + D^{-2} \circ a_{-2} + \cdots\,.$$

Example: Let

$$A = uD^2 + u_1 D, \qquad B = -u_1 D^3, \qquad C = uD^{-1}$$

then

$$A^+ = D^2 \circ u - D \circ u_1 = A, \qquad B^+ = D^3 \circ u_1 = u_1 D^3 + 3u_2 D^2 + 3u_3 D + u_4\,,$$

$$C^+ = -D^{-1} \circ u = -uD^{-1} + u_1 D^{-2} - u_2 D^{-3} + \cdots\,.$$

Formal series form a **skew-field**. For any element (23) we can find uniquely the inverse element

$$B = b_{-m} D^{-m} + b_{-m-1} D^{-m-1} + \cdots\,, \quad b_k \in \mathfrak{F}$$

such that $A \circ B = B \circ A = 1$. Indeed, in $A \circ B = 1$

$$\begin{aligned}
&\text{at} \quad D^0: \quad a_m b_{-m} = 1, \ \Rightarrow \ b_{-m} = 1/a_m \\
&\text{at} \quad D^{-1}: \quad m a_m D(b_{-m}) + a_m b_{-m-1} + a_{m-1} b_{-m} = 0 \\
&\qquad \Rightarrow \quad b_{-m-1} = -\frac{a_{m-1}}{a_m^2} - mD(\frac{1}{a_m}), \text{etc.}
\end{aligned}$$

First k coefficients of the series B can be uniquely determined in terms of the first k coefficients of A.

Moreover we can find the m-th root of the series A (23), i.e. a series

$$C = c_1 D + c_0 + c_{-1} D^{-1} + c_{-2} D^{-2} + \cdots , \qquad c_k \in \mathfrak{F}$$

such that $C^m = A$. The series C is unique up to a constant factor ω, $\omega^m = 1$. Coefficients of C can be found recursively.

Example: Let $A = D^2 + u$. Assuming $C = c_1 D + c_0 + c_{-1} D^{-1} + c_{-2} D^{-2} + \cdots$ we get:

$$C^2 = C \circ C = (c_1 D + c_0 + c_{-1} D^{-1} + \cdots) \circ (c_1 D + c_0 + c_{-1} D^{-1} + \cdots) =$$

$$c_1^2 D^2 + (c_1 D(c_1) + c_1 c_0 + c_0 c_1) D + c_1 D(c_0) + c_0^2 + c_1 c_{-1} + c_{-1} c_1 + \cdots ,$$

and compare the result with A. At D^2 we find $c_1^2 = 1$ or $c_1 = \pm 1$. Let we choose the positive root $c_1 = 1$. Now at D we have $2c_0 = 0$, i.e. $c_0 = 0$. At D^0 we have $2c_{-1} = u$, at D^{-1} we find $c_{-2} = -u_1/4$, etc.,

$$C = D + \frac{u}{2} D^{-1} - \frac{u_1}{4} D^{-2} + \cdots .$$

We can easily find as many coefficients of C as required.

Definition 4. *The residue of a formal series $A=\sum_{k\leq n} a_k D^k,\ a_k \in \mathfrak{F}$ is the coefficient at D^{-1}*

$$\textit{res}\,(A) = a_{-1}\,.$$

The logarithmic residue of A is defined as

$$\textit{res}\,\log A = \frac{a_{n-1}}{a_n}\,.$$

For any two formal series A, B of order n and m respectively the logarithmic residue satisfies the following identity

$$\text{res}\,\log(A \circ B) = \text{res}\,\log(A) + \text{res}\,\log(B) + nD(\log(b_m))\,.$$

For any derivation D_t of the field $\mathfrak{F}$ and any formal series A we have

$$D_t(\,\text{res}\,\log(A)) = \text{res}\,(D_t(A) \circ A^{-1})\,. \tag{25}$$

Let C be the n-th root of A, i.e. $C^n = A$. We define a sequence

$$\rho_{-1} = \text{res}\,C^{-1}, \rho_0 = \text{res}\,\log C, \rho_1 = \text{res}\,C, \ldots, \rho_k = \text{res}\,C^k, \ldots.$$

Then the first m elements of this sequence are in one-to-one correspondence with the first m coefficients of C.

We will use the following important Adler's Theorem (M.Adler, Inventiones Math., 50, 219-248, 1979)

Theorem 2. *For any two formal series A, B the residue of the commutator belongs to Im(D):*

$$\textit{res}\,[A,B] = D(\sigma(A,B)), \qquad \sigma(A,B) \in \mathfrak{F}\,.$$

Proof: One can readily show that

$$\sigma(A,B) = \sum_{p\leq \mathrm{ord}(B),\ q\leq \mathrm{ord}(A)}^{p+q+1>0} C_q^{p+q+1} \sum_{s=0}^{p+q} (-1)^s D^s(a_q) D^{p+q-s}(b_p)\,.$$

Formal recursion operator

For simplicity here and in the sequel we assume that $\mathfrak{F} = \mathfrak{F}(x, u; D)$, i.e. the right-hand side of equation

$$u_t = F(u, \dots, u_n, x)\,, \qquad n \geq 2 \tag{26}$$

and also its symmetries and conservation laws do not depend on t explicitly.

Definition 5. *A formal series*

$$\Lambda = l_1 D + l_0 + l_{-1} D^{-1} + \cdots\,, \qquad l_k \in \mathfrak{F} \tag{27}$$

is called a **formal recursion operator** *for equation (26) if it satisfies equation*

$$D_t(\Lambda) - [F_*, \Lambda] = 0\,. \tag{28}$$

Important Remark We will never consider Λ as an operator, i.e. we will never act by Λ to any function. Relation (28) is regarded as an infinite system of equations for the coefficients l_i.

Proposition 1. *If Λ is a formal recursion operator for equation (26), then any power $\hat{\Lambda} = \Lambda^k$ also satisfy equation (28). In particular,*

$$\tilde{\Lambda} = c_1\Lambda + c_0 + c_{-1}\Lambda^{-1} + c_{-2}\Lambda^{-2} + \cdots$$

is a formal recursion operator for (26) for any $c_k \in \mathbb{C}$.

If $\hat{\Lambda}$ is a psudo-differential operator satisfying equation (28) and its action is properly defined on a symmetry generator G (i.e. $\hat{\Lambda}(G) \in \mathfrak{F}$), then $H = \hat{\Lambda}(G)$ is a generator of a symmetry.

$$D_t(H) = D_t(\hat{\Lambda}G) = D_t(\hat{\Lambda})(G) + \hat{\Lambda}D_t(G) =$$

$$= [F_*, \hat{\Lambda}](G) + \hat{\Lambda}F_*(G) = F_*\hat{\Lambda}(G) = F_*(H).$$

In this case Λ is called a *recursion operator*.

Example. The Korteweg-de Vries equation $u_t = u_3 + 6uu_1$ has a recursion operator

$$\widehat{\Lambda} = D^2 + 4u + 2u_1 D^{-1}\,,$$

which satisfies equation (28). The formal recursion operator for the Korteweg -de Vries equation can be represented as $\Lambda = \widehat{\Lambda}^{1/2}$. The infinite hierarchy of commutative symmetries of KdV can be obtained as

$$G_{2k+1} = \widehat{\Lambda}^k(u_1)\,.$$

Example: The Burgers equation $u_t = u_2 + 2uu_1$ has the (formal) recursion operator

$$\Lambda = D + u + u_1 D^{-1}\,.$$

Functions $G_n = \Lambda^n(u_1)$ are generators of symmetries of the Burgers equation.

The concept of formal recursion operator is very universal in the theory of integrable equations. A formal series Λ satisfying equation (28) exists and the sequence of its coefficients $l_1, l_0, ... \in \mathfrak{F}$ can be found explicitly if equation (26) possesses an infinite hierarchy of symmetries or conservation laws of arbitrary high order, or can be linearized by a differential substitution.

Theorem 3. *If equation $u_t = F,\ F \in \mathfrak{F}$ possesses an infinite hierarchy of higher symmetries of infinitely increasing order then it has a formal recursion operator.*

The main idea of the proof of this Theorem and the relation between the structure of the formal recursion operator and symmetries can be illustrated by the following consideration. Suppose equation (26) has a symmetry with a generator G. Function G satisfies equation (21). Let us compute the Fréchet derivative from this equation. Using identities (13),(15),(**??**) we get equation

$$D_t(G_*) + G_* F_* = D_G(F_*) + F_* G_*$$

which can be rearrange in the form

$$D_t(G_*) - [F_*, G_*] = D_G(F_*)\,. \tag{29}$$

Now let us assume that the order of equation (26) is fixed, say $n = 3$ (i.e. $F = F(u, u_1, u_2, u_3, x)$ in (26)), and the symmetry G has a very high order (say, for example, $m = 125$, i.e. $G = G(u, u_1, ..., u_{125}, x)$). Equation (29) for operators (the Fréchet derivative is a differential operator) is understood as equations for the coefficients of the operators at each power D^k. In the right-hand side of equation (29) we have operator $D_G(F_*)$ of order 3 (or less, if the leading coefficient of F_* is a constant). The product $F_* G_*$ in the left-hand side of the equation has the order $n + m = 3 + 125 = 128$. It means that in first $128 - 3 = 125$ equations the right-hand side does not contribute and first 124 terms of operator G_* satisfy the same equation (28) as the formal recursion operator Λ. We can use $G_*^{1/125}$ as an approximate for Λ or, more precisely,

$$\Lambda = (G_*)^{1/m} + \tilde{l}_{-123} D^{-123} + \tilde{l}_{-124} D^{-124} + \cdots . \qquad (30)$$

If equation (26) has an infinite hierarchy of symmetries G_s and the order of symmetries is going to infinity as $s \to \infty$ then one can show that there exist a formal series Λ, such that equation (28) is satisfied at any order D^k, $k = n, n - 1, ..., 0, -1, ...$. That is the basic idea for the proof of the Theorem.