

# NLEVP: A Collection of Nonlinear Eigenvalue Problems

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We present a collection of 52 nonlinear eigenvalue problems in the form of a MATLAB toolbox. The collection contains problems from models of real-life applications as well as ones constructed specifically to have particular properties. A classification is given of polynomial eigenvalue problems according to their structural properties. Identifiers based on these and other properties can be used to extract particular types of problems from the collection. A brief description of each problem is given. NLEVP serves both to illustrate the tremendous variety of applications of nonlinear eigenvalue problems and to provide representative problems for testing, tuning, and benchmarking of algorithms and codes.

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## 1. INTRODUCTION

In many areas of scientific computing collections of problems are available that play an important role in developing algorithms and in testing and benchmarking software. Among the uses of such collections are

- tuning an algorithm to optimize its performance across a wide and representative range of problems;
- testing the correctness of a code against some measure of success, where the latter is typically an error or residual whose nature is suggested by the underlying problem;
- measuring the performance of a code—for example, speed, execution rate, or again an error or residual;
- measuring the robustness of a code, that is, the behaviour in extreme situations, such as for very badly scaled and/or ill conditioned data;
- comparing two or more codes with respect to the factors above.

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A collection ideally combines problems artificially constructed to reflect a wide range of possible properties with problems representative of real applications. Problems for which something is known about the solution are always particularly attractive.

The practice of reproducible research, whereby research is published in such a way that the underlying numerical (and other) experiments can be repeated by others, is of growing interest and visibility [Donoho et al. 2009], [LeVeque 2009], [Mesirov 2010]. Reproducible research is aided by the availability of well documented and maintained benchmark collections.

Two areas that have historically been well endowed with collections of problems implemented in software are linear algebra and optimization. In linear algebra an early collection is ACM Algorithm 694 [Higham 1991], which contains parametrized, mainly dense, test matrices, most of which were later incorporated into the MATLAB gallery function. The University of Florida Sparse Matrix Collection is a regularly updated collection of sparse matrices [Davis], [Davis and Hu 2011], with over 2200 matrices from practical applications. It includes all the matrices from the earlier Matrix Market repository (although not the matrix generators) [Matrix Market], the Harwell–Boeing collection [Duff et al. 1989] of sparse matrices, and the NEP collection [Bai et al. 1997] of standard and generalized eigenvalue problems. The CONTEST toolbox [Taylor and Higham 2009] produces adjacency matrices describing random networks. In optimization we mention just the collections in the widely used Cute and Cuter testing environments [Bongartz et al. 1995], [Gould et al. 2003], though various other, sometimes more specialized, collections are available.

The growing interest in nonlinear eigenvalue problems has created the need for a collection of problems in this area. The standard form of a nonlinear eigenvalue problem is  $F(\lambda)x = 0$ , where  $F : \mathbb{C} \rightarrow \mathbb{C}^{m \times n}$  is a given matrix-valued function and  $\lambda \in \mathbb{C}$  and the nonzero vector  $x \in \mathbb{C}^n$  are the sought eigenvalue and eigenvector, respectively. Rational and polynomial functions are of particular interest, the most practically important case being the quadratic  $Q(\lambda) = \lambda^2 A + \lambda B + C$ , which corresponds to the quadratic eigenvalue problem. For recent surveys on nonlinear eigenproblems see [Mehrmann and Voss 2004] and [Tisseur and Meerbergen 2001]. Associated with an  $n \times n$  matrix quadratic  $Q(\lambda)$  are the matrix equations  $X^2 A + X B + C = 0$  and  $A X^2 + B X + C = 0$ , where the unknown  $X \in \mathbb{C}^{n \times n}$  is called a solvent [Dennis et al. 1976], [Gohberg et al. 2009] [Higham and Kim 2000]. Thus a matrix polynomial  $P(\lambda)$  defines both an eigenvalue problem and two matrix equations.

We have built a collection of nonlinear eigenvalue problems from a variety of sources. Some are from models of real-life applications, while others have been constructed specifically to have particular properties. Many of the matrices have been used in previous papers to test numerical algorithms. In order to provide focus and keep the collection to a manageable size we have chosen to exclude linear problems from the collection. The problems range from the old, such as the wing problem from the classic 1938 book of Frazer, Duncan, and Collar [Frazer et al. 1938], to the very recent, notably several problems from research in 3D vision that are not yet well known in the numerical analysis community.

Nonlinear eigenvalue problems are often highly structured and it is important to take account of the structure both in developing the theory and in designing numerical methods. We therefore provide a thorough classification of our problems that records the most relevant structural properties.

We have chosen to implement the collection in MATLAB, as a toolbox, recognizing that it is straightforward to convert the matrices into a format that can be read by other languages by using either the built-in MATLAB I/O functions or those provided in Matrix Market. Care has been taken to make the toolbox compatible with GNU Octave [GNU Octave]. A criterion for inclusion of problems is that the underlying MAT-

LAB code and data files are not too large, since we want to provide the toolbox as a single file that can be downloaded in a reasonable time.

The NLEVP toolbox is available, as both a zip file and a tar file, from

<http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html>

For details of how to install and use the toolbox see [Bettcke et al. 2011].

In Section 2 we explain how we classify the problems through identifiers that can be used to extract specific types of problem from the collection. The main features of the problems are described in Section 3, while Section 4 describes the design of the toolbox. Conclusions are given in Section 5.

## 2. IDENTIFIERS

We give in Table I a list of identifiers for the types of problems available in the collection and in Table II a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. All the identifiers are case insensitive. In the next two subsections we briefly recall some relevant definitions and properties of nonlinear eigenproblems.

### 2.1. Nonlinear Eigenproblems

The **polynomial eigenvalue problem** (PEP) is to find scalars  $\lambda$  and nonzero vectors  $x$  and  $y$  satisfying  $P(\lambda)x = 0$  and  $y^*P(\lambda) = 0$ , where

$$P(\lambda) = \sum_{i=0}^k \lambda^i A_i, \quad A_i \in \mathbb{C}^{m \times n}, \quad A_k \neq 0 \quad (1)$$

is an  $m \times n$  matrix polynomial of degree  $k$ . Here,  $x$  and  $y$  are right and left eigenvectors corresponding to the eigenvalue  $\lambda$ . The **reversal** of the matrix polynomial (1) is defined by

$$\text{rev}(P(\lambda)) = \lambda^k P(1/\lambda) = \sum_{i=0}^k \lambda^{k-i} A_i.$$

A PEP is said to have an eigenvalue  $\infty$  if zero is an eigenvalue of  $\text{rev}(P(\lambda))$ .

A **quadratic eigenvalue problem** (QEP) is a PEP of degree  $k = 2$ . For a survey of QEPs see [Tisseur and Meerbergen 2001]. Polynomial and quadratic eigenproblems are identified by pep and qep, respectively, in the collection (see Table I), and any problem of type qep is automatically also of type pep.

The matrix function  $R(\lambda) \in \mathbb{C}^{m \times n}$  whose elements are rational functions

$$r_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{q_{ij}(\lambda)}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

where  $p_{ij}(\lambda)$  and  $q_{ij}(\lambda)$  are scalar polynomials of the same variable and  $q_{ij}(\lambda) \not\equiv 0$ , defines a **rational eigenvalue problem** (REP)  $R(\lambda)x = 0$  [Kublanovskaya 1999]. Unlike for PEPs there is no standard format for specifying REPs. For the collection we use the form

$$R(\lambda) = P(\lambda)Q(\lambda)^{-1},$$

where  $P(\lambda)$  and  $Q(\lambda)$  are matrix polynomials, or the less general form (often encountered in practice)

$$R(\lambda) = A + \lambda B + \sum_{i=1}^{k-1} \frac{\lambda}{\sigma_i - \lambda} C_i, \quad (2)$$

Table I. Problems available in the collection and their identifiers.

qep	quadratic eigenvalue problem
pep	polynomial eigenvalue problem
rep	rational eigenvalue problem
nep	other nonlinear eigenvalue problem

Table II. List of identifiers for the problem properties.

nonregular	symmetric	hyperbolic
real	hermitian	elliptic
nonsquare	T-even	overdamped
sparse	*-even	proportionally-damped
scalable	T-odd	gyroscopic
parameter-dependent	*-odd	
solution	T-palindromic	
random	*-palindromic	
	T-anti-palindromic	
	*-anti-palindromic	

where  $A$ ,  $B$ , and the  $C_i$  are  $m \times n$  matrices, and the  $\sigma_i$  are the poles. Which form is used is specified in the help for the M-file defining the problem. Rational eigenproblems are identified by rep in the collection.

As mentioned in the introduction, PEPs and REPs are special cases of **nonlinear eigenvalue problems** (NEPs)  $F(\lambda)x = 0$ , where  $F : \mathbb{C} \rightarrow \mathbb{C}^{m \times n}$ . A convenient general form for expressing an NEP is

$$F(\lambda) = \sum_{i=0}^k f_i(\lambda) A_i, \quad (3)$$

where the  $f_i : \mathbb{C} \rightarrow \mathbb{C}$  are nonlinear functions and  $A_i \in \mathbb{C}^{m \times n}$ . Any problem that is not polynomial, quadratic, or rational is identified by nep in the collection (see Table I).

## 2.2. Some Definitions and Properties

Nonlinear eigenproblems are said to be **regular** if  $m = n$  and  $\det(F(\lambda)) \not\equiv 0$ , and **non-regular** otherwise. Recall that a regular PEP possesses  $nk$  (not necessarily distinct) eigenvalues [Gohberg et al. 2009], including infinite eigenvalues. As the majority of problems in the collection are regular we identify only nonregular problems, for which the identifier is nonregular.

The identifiers real, hermitian, and symmetric are defined in Table III. For PEPs, the real identifier corresponds to  $P$  having real coefficient matrices, while hermitian corresponds to Hermitian (but not all real) coefficient matrices. Similarly, symmetric indicates (complex) symmetric coefficient matrices, and the real identifier is added if the coefficient matrices are real symmetric. For problems that are parameter-dependent the identifiers real and hermitian are used if the problem is real or Hermitian for real values of the parameter.

Table III. Some identifiers and the corresponding spectral properties. For parameter-dependent problems, the problem is classified as *real* or *hermitian* if it is so for real values of the parameter.

Identifier	Property of $F(\lambda) \in \mathbb{C}^{m \times n}$	Spectral properties
<i>real</i>	$\overline{F(\lambda)} = F(\bar{\lambda})$	eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$
<i>symmetric</i>	$m = n, (F(\lambda))^T = F(\lambda)$	none unless $F$ is real
<i>hermitian</i>	$m = n, (F(\lambda))^* = F(\bar{\lambda})$	eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$

Table IV. Some identifiers and the corresponding spectral symmetry properties.

Identifier	Property of $P(\lambda)$	Eigenvalue pairing
<i>T-even</i>	$P^T(-\lambda) = P(\lambda)$	$(\lambda, -\lambda)$
<i>*-even</i>	$P^*(-\lambda) = P(\lambda)$	$(\lambda, -\bar{\lambda})$
<i>T-odd</i>	$P^T(-\lambda) = -P(\lambda)$	$(\lambda, -\lambda)$
<i>*-odd</i>	$P^*(-\lambda) = -P(\lambda)$	$(\lambda, -\bar{\lambda})$
<i>T-palindromic</i>	$\text{rev}P^T(\lambda) = P(\lambda)$	$(\lambda, 1/\lambda)$
<i>*-palindromic</i>	$\text{rev}P^*(\lambda) = P(\lambda)$	$(\lambda, 1/\bar{\lambda})$
<i>T-anti-palindromic</i>	$\text{rev}P^T(\lambda) = -P(\lambda)$	$(\lambda, 1/\lambda)$
<i>*-anti-palindromic</i>	$\text{rev}P^*(\lambda) = -P(\lambda)$	$(\lambda, 1/\bar{\lambda})$

Definitions of identifiers for odd-even and palindromic-like square matrix polynomials, together with the special symmetry properties of their spectra (see [Mackey et al. 2006]) are given in Table IV.

**Gyroscopic** systems of the form  $Q(\lambda) = \lambda^2 M + \lambda G + K$  with  $M, K$  Hermitian,  $M > 0$ , and  $G = -G^*$  skew-Hermitian are a subset of *\*-even* (*T-even* when the coefficient matrices are real) QEPs and are identified with *gyroscopic*. Here, for a Hermitian matrix  $A$ , we write  $A > 0$  to denote that  $A$  is positive definite and  $A \geq 0$  to denote that  $A$  is positive semidefinite. When  $K > 0$  the eigenvalues of  $Q$  are purely imaginary and semisimple [Duffin 1960], [Lancaster 1966] and the quadratic  $Q(i\lambda)$  is hyperbolic.

A Hermitian matrix polynomial  $P(\lambda)$  is **hyperbolic** if there exists  $\mu \in \mathbb{R} \cup \{\infty\}$  such that  $P(\mu)$  is positive definite and for every nonzero  $x \in \mathbb{C}^n$  the scalar equation  $x^* P(\lambda) x = 0$  has  $k$  distinct zeros in  $\mathbb{R} \cup \{\infty\}$ . All the eigenvalues of such a  $P$  are real, semisimple, and grouped in  $k$  intervals, each of them containing  $n$  eigenvalues [Al-Ammari and Tisseur 2011], [Higham et al. 2009], [Markus 1988]. These polynomials are identified in the collection by *hyperbolic*. **Overdamped** systems  $Q(\lambda) = \lambda^2 M + \lambda C + K$  are particular hyperbolic QEPs for which  $M > 0$ ,  $C > 0$ , and  $K \geq 0$ ; they have the identifier *overdamped*. Finally, a QEP is said to be **proportionally damped** when  $M, C$ , and  $K$  are simultaneously diagonalizable by congruence or strict equivalence [Lancaster and Zaballa 2009] (a sufficient condition for which is that  $C = \alpha M + \beta K$  with  $M$  and  $K$  simultaneously diagonalizable, hence the name), and such a QEP is identified by *proportionally-damped*.

Hermitian matrix polynomials  $P(\lambda)$  with even degree  $k$  that are **elliptic**, i.e.,  $P(\lambda) > 0$  for all  $\lambda \in \mathbb{R}$  [Markus 1988, §34], are identified by *elliptic*. Elliptic matrix polynomials have nonreal eigenvalues.

The identifier *sparse* is used if the defining matrices are stored in the MATLAB sparse format. Problems that depend on one or more parameters are identified with *parameter-dependent*. Problems for which random numbers are used in the construc-

Table V. Quadratic eigenvalue problems.

acoustic_wave_1d	acoustic_wave_2d	bicycle	bilby
cd_player	closed_loop	concrete	damped_beam
dirac	foundation	gen_hyper2	gen_tantipal2
gen_tpal2	intersection	hospital	metal_strip
mobile_manipulator	omnicam1	omnicam2	pdde_stability
power_plant	qep1	qep2	qep3
qep4	qep5	railtrack	railtrack2
relative_pose_6pt	schrodinger	shaft	sign1
sign2	sleeper	speaker_box	spring
spring_dashpot	surveillance	wing	wiresaw1
wiresaw2			

Table VI. Other eigenvalue problems.

Polynomial, degree $\geq 3$	butterfly	mirror	orr_sommerfeld
Nonsquare polynomial	planar_waveguide	plasma_drift	relative_pose_5pt
Nonregular polynomial	qep4	surveillance	
Rational	qep4	qep5	surveillance
Nonlinear	loaded_string		
	fiber	gun	haderler
	time_delay		

tion are identified with random. A separate identifier, *scalable*, is used to denote that the problem dimension (or an approximation of it) is a parameter. For parameter-dependent problems a default value of the parameter is provided, typically being a value used in previously published experiments.

For some problems a *supposed* solution is optionally returned, comprising eigenvalues and/or eigenvectors that are exactly known, approximate, or computed. These problems are identified with *solution*. The documentation for the problem provides information on the nature of the supposed solution.

Tables V and VI identify the QEPs, the PEPs that are of degree at least 3, the non-square PEPs, the REPs, and the nonlinear but non-polynomial and non-rational problems in the collection.

### 3. COLLECTION OF PROBLEMS

This section contains a brief description of all the problems in the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem. The problems are summarized in Table VII.

We use the following notation.  $A \otimes B$  denotes the Kronecker product of  $A$  and  $B$ , namely the block matrix  $(a_{ij}B)$  [Higham 2008, Sec. B.13]. The  $i$ th unit vector (that is, the  $i$ th column of the identity matrix) is denoted by  $e_i$ .

`acoustic_wave_1d` {pep,qep,symmetric,\*-even,parameter-dependent,sparse,scalable}.

This quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$  arises from the finite element discretization of the time-harmonic wave equation  $-\Delta p - (2\pi f/c)^2 p = 0$  for the acoustic pressure  $p$  in a bounded domain, where the boundary conditions are partly Dirichlet ( $p = 0$ ) and partly impedance ( $\frac{\partial p}{\partial n} + \frac{2\pi i f}{\zeta} p = 0$ ) [Chaitin-Chatelin and van Gijzen 2006]. Here,  $f$  is the frequency,  $c$  is the speed of sound in the medium, and  $\zeta$  is the (possibly complex) impedance. We take  $c = 1$  as in [Chaitin-Chatelin and van Gijzen 2006]. The eigenvalues of  $Q$  are the resonant frequencies of the system, and

Table VII. Problems in NLEVP.

acoustic_wave_1d	Acoustic wave problem in 1 dimension.
acoustic_wave_2d	Acoustic wave problem in 2 dimensions.
bicycle	2-by-2 QEP from the Whipple bicycle model.
bilby	5-by-5 QEP from bilby population model.
butterfly	Quartic matrix polynomial with T-even structure.
cd_player	QEP from model of CD player.
closed_loop	2-by-2 QEP associated with closed-loop control system.
concrete	Sparse QEP from model of a concrete structure.
damped_beam	QEP from simply supported beam damped in the middle.
dirac	QEP from Dirac operator.
fiber	NEP from fiber optic design.
foundation	Sparse QEP from model of machine foundations.
gen_hyper2	Hyperbolic QEP constructed from prescribed eigenpairs.
gen_tantipal2	T-anti-palindromic QEP with eigenvalues on the unit circle.
gen_tpal2	T-palindromic QEP with prescribed eigenvalues on the unit circle.
gun	NEP from model of a radio-frequency gun cavity.
hadeler	NEP due to Hadeler.
intersection	10-by-10 QEP from intersection of three surfaces.
hospital	QEP from model of Los Angeles Hospital building.
loaded_string	REP from finite element model of a loaded vibrating string.
metal_strip	QEP related to stability of electronic model of metal strip.
mirror	Quartic PEP from calibration of cadioptric vision system.
mobile_manipulator	QEP from model of 2-dimensional 3-link mobile manipulator.
omnicam1	9-by-9 QEP from model of omnidirectional camera.
omnicam2	15-by-15 QEP from model of omnidirectional camera.
orr_sommerfeld	Quartic PEP arising from Orr-Sommerfeld equation.
pdde_stability	QEP from stability analysis of discretized PDDE.
planar_waveguide	Quartic PEP from planar waveguide.
plasma_drift	Cubic PEP arising in Tokamak reactor design.
power_plant	8-by-8 QEP from simplified nuclear power plant problem.
qep1	3-by-3 QEP with known eigensystem.
qep2	3-by-3 QEP with known, nontrivial Jordan structure.
qep3	3-by-3 parametrized QEP with known eigensystem.
qep4	3-by-4 QEP with known, nontrivial Jordan structure.
qep5	3-by-3 nonregular QEP with known Smith form.
railtrack	QEP from study of vibration of rail tracks.
railtrack2	Palindromic QEP from model of rail tracks.
relative_pose_5pt	Cubic PEP from relative pose problem in computer vision.
relative_pose_6pt	QEP from relative pose problem in computer vision.
schrodinger	QEP from Schrodinger operator.
shaft	QEP from model of a shaft on bearing supports with a damper.
sign1	QEP from rank-1 perturbation of sign operator.
sign2	QEP from rank-1 perturbation of $2*\sin(x) + \text{sign}(x)$ operator.
sleeper	QEP modelling a railtrack resting on sleepers.
speaker_box	QEP from model of a speaker box.
spring	QEP from finite element model of damped mass-spring system.
spring_dashpot	QEP from model of spring/dashpot configuration.
surveillance	21-by-16 QEP from surveillance camera callibration.
time_delay	3-by-3 NEP from time-delay system.
wing	3-by-3 QEP from analysis of oscillations of a wing in an airstream.
wiresaw1	Gyroscopic QEP from vibration analysis of a wiresaw.
wiresaw2	QEP from vibration analysis of wiresaw with viscous damping.

for the given problem formulation they lie in the upper half of the complex plane. For more on the discretization of acoustics problems see, for example, [Harari et al. 1996].

On the 1D domain  $[0, 1]$  the  $n \times n$  matrices are defined by

$$M = -4\pi^2 \frac{1}{n} (I_n - \frac{1}{2} e_n e_n^T), \quad C = 2\pi i \frac{1}{\zeta} e_n e_n^T, \quad K = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}.$$

`acoustic_wave_2d` {pep,qep,symmetric,\*-even,parameter-dependent,sparse,scalable}.

A 2D version of Acoustic wave 1D. On the unit square  $[0, 1] \times [0, 1]$  with mesh size  $h$  the  $n \times n$  coefficient matrices of  $Q(\lambda)$  with  $n = \frac{1}{h}(\frac{1}{h} - 1)$  are given by

$$M = -4\pi^2 h^2 I_{m-1} \otimes (I_m - \frac{1}{2} e_m e_m^T), \quad D = 2\pi i \frac{h}{\zeta} I_{m-1} \otimes (e_m e_m^T), \\ K = I_{m-1} \otimes D_m + T_{m-1} \otimes (-I_m + \frac{1}{2} e_m e_m^T),$$

where  $\otimes$  denotes the Kronecker product,  $m = 1/h$ ,  $\zeta$  is the (possibly complex) impedance, and

$$D_m = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 4 & -1 \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{m \times m}, \quad T_{m-1} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(m-1) \times (m-1)}.$$

The eigenvalues of  $Q$  are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane.

`bicycle` {pep,qep,real,parameter-dependent}. This is a  $2 \times 2$  quadratic polynomial arising in the study of bicycle self-stability [Meijaard et al. 2007]. The linearized equations of motion for the Whipple bicycle model can be written as

$$M\ddot{q} + C\dot{q} + Kq = f,$$

where  $M$  is a symmetric mass matrix, the nonsymmetric damping matrix  $C = vC_1$  is linear in the forward speed  $v$ , and the stiffness matrix  $K = gK_0 + v^2K_2$  is the sum of two parts: a velocity independent symmetric part  $gK_0$  proportional to the gravitational acceleration  $g$  and a nonsymmetric part  $v^2K_2$  quadratic in the forward speed.

`bilby` {pep,qep,real,parameter-dependent}. This  $5 \times 5$  quadratic matrix polynomial arises in a model from [Bean et al. 1997] for the population of the greater bilby (*Macrotis lagotis*), an endangered Australian marsupial. Define the  $5 \times 5$  matrix

$$M(g, x) = \begin{bmatrix} gx_1 & (1-g)x_1 & 0 & 0 & 0 \\ gx_2 & 0 & (1-g)x_2 & 0 & 0 \\ gx_3 & 0 & 0 & (1-g)x_3 & 0 \\ gx_4 & 0 & 0 & 0 & (1-g)x_4 \\ gx_5 & 0 & 0 & 0 & (1-g)x_5 \end{bmatrix}.$$

The model is a quasi-birth-death process some of whose key properties are captured by the elementwise minimal solution of the quadratic matrix equation

$$R = \beta(A_0 + RA_1 + R^2A_2), \quad A_0 = M(g, b), \quad A_1 = M(g, e - b - d), \quad A_2 = M(g, d),$$

where  $b$  and  $d$  are vectors of probabilities and  $e$  is the vector of ones. The corresponding quadratic matrix polynomial is  $Q(\lambda) = \lambda^2A + \lambda B + C$ , where

$$A = \beta A_2^T, \quad B = \beta A_1^T - I, \quad C = \beta A_0^T.$$



We take  $g = 0.2$ ,  $b = [1, 0.4, 0.25, 0.1, 0]^T$ , and  $d = [0, 0.5, 0.55, 0.8, 1]^T$ , as in [Bean et al. 1997].

`butterfly` {pep,real,parameter-dependent,T-even,sparse,scalable}. This is a quartic matrix polynomial  $P(\lambda) = \lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$  of dimension  $m^2$  with T-even structure, depending on a  $10 \times 1$  parameter vector  $c$  [Mehrmann and Watkins 2002]. Its spectrum has a butterfly shape. The coefficient matrices are Kronecker products, with  $A_4$  and  $A_2$  real and symmetric and  $A_3$  and  $A_1$  real and skew-symmetric, assuming  $c$  is real. The default is  $m = 8$ .

`cd_player` {pep,qep,real}. This is a  $60 \times 60$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$ , with  $M = I_{60}$  arising in the study of a CD player control task [Chahlaoui and Van Dooren 2002], [Chahlaoui and Van Dooren 2005], [Draijer et al. 1992], [Wortelboer et al. 1996]. The mechanism that is modeled consists of a swing arm on which a lens is mounted by means of two horizontal leaf springs. This is a small representation of a larger original rigid body model (which is also quadratic).

`closed_loop` {pep,qep,real,parameter-dependent}. This is a quadratic polynomial

$$Q(\lambda) = \lambda^2 I + \lambda \begin{bmatrix} 0 & 1 + \alpha \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

associated with a closed-loop control system with feedback gains 1 and  $1 + \alpha$ ,  $\alpha \geq 0$ . The eigenvalues of  $Q(\lambda)$  lie inside the unit disc if and only if  $0 \leq \alpha < 0.875$  [Tisseur and Higham 2001].

`concrete` {pep,qep,symmetric,parameter-dependent,sparse}. This is a quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + (1 + i\mu)K$  arising in a model of a concrete structure supporting a machine assembly [Feriani et al. 2000]. The matrices have dimension 2472.  $M$  is real diagonal and low rank.  $C$ , the viscous damping matrix, is pure imaginary and diagonal.  $K$  is complex symmetric, and the factor  $1 + i\mu$  adds uniform hysteretic damping. The default is  $\mu = 0.04$ .

`damped_beam` {pep,qep,real,symmetric,sparse,scalable}. This QEP arises in the vibration analysis of a beam simply supported at both ends and damped in the middle [Higham et al. 2008]. The quadratic  $Q(\lambda) = \lambda^2 M + \lambda C + K$  has real symmetric coefficient matrices with  $M > 0$ ,  $K > 0$ , and  $C = ce_n e_n^T \geq 0$ , where  $c$  is a damping parameter. Half of the eigenvalues of the problem are purely imaginary and are eigenvalues of the undamped problem ( $C = 0$ ).

`dirac` {pep,qep,real,symmetric,parameter-dependent,scalable}. The spectrum of this matrix polynomial is the second order spectrum of the radial Dirac operator with an electric Coulombic potential of strength  $\alpha$ ,

$$D = \begin{bmatrix} 1 + \frac{\alpha}{r} & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -1 + \frac{\alpha}{r} \end{bmatrix}.$$

For  $-\sqrt{3}/2 < \alpha < 0$  and  $\kappa \in \mathbb{Z}$ ,  $D$  acts on  $L^2((0, \infty), \mathbb{C}^2)$  and it corresponds to a spherically symmetric decomposition of the space into partial wave subspaces [Thaller 1992]. The problem discretization is relative to subspaces generated by the Hermite functions of odd order. The size of the matrix coefficients of the QEP is  $n + m$ , corresponding to  $n$  Hermite functions in the first component of the  $L^2$  space and  $m$  in the second component [Boulton and Boussaid 2010].

For  $\kappa = -1$ ,  $\alpha = -1/2$  and  $n$  large enough, there is a conjugate pair of isolated points of the second order spectrum near the ground eigenvalue  $E_0 \approx 0.866025$ . The essential

spectrum,  $(-\infty, -1] \cup [1, \infty)$ , as well as other eigenvalues, also seem to be captured for large  $n$ .

`fiber` {nep,sparse,solution}. This nonlinear eigenvalue problem arises from a model in fiber optic design based on the Maxwell equations [Huang et al. 2010], [Kaufman 2006]. The problem is of the form

$$F(\lambda)x = (A - \lambda I + s(\lambda)B)x = 0,$$

where  $A \in \mathbb{R}^{2400 \times 2400}$  is tridiagonal and  $B = e_{2400} e_{2400}^T$ . The scalar function  $s(\lambda)$  is defined in terms of Bessel functions. The real, positive eigenvalues are the ones of interest.

`foundation` {pep,qep,symmetric,sparse}. This is a quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$  arising in a model of reinforced concrete machine foundations resting on the ground [Feriani et al. 2000]. The matrices have dimension 3627;  $M$  is real and diagonal,  $C$  is complex and diagonal, and  $K$  is complex symmetric.

`gen_hyper2` {pep,qep,real,symmetric,hyperbolic,parameter-dependent,scalable,solution,random}. This is a hyperbolic quadratic matrix polynomial generated from a given set of eigenvalues and eigenvectors  $(\lambda_k, v_k)$ ,  $k = 1: 2n$ , such that with

$$\begin{aligned} A &= \text{diag}(\lambda_1, \dots, \lambda_{2n}) =: \text{diag}(A_1, A_2), & A_1, A_2 &\in \mathbb{R}^{n \times n}, \\ V &:= [v_1, \dots, v_{2n}] =: [V_1 \quad V_2], & V_1, V_2 &\in \mathbb{R}^{n \times n}, \end{aligned}$$

$\lambda_{\min}(A_1) > \lambda_{\max}(A_2)$ ,  $V_1$  is nonsingular, and  $V_2 = V_1 U$  for some orthogonal matrix  $U$ . Then the  $n \times n$  symmetric quadratic  $Q(\lambda) = \lambda^2 A + \lambda B + C$  with

$$\begin{aligned} A &= \Gamma^{-1}, & \Gamma &= V_1 A_1 V_1^T - V_2 A_2 V_2^T, \\ B &= -A(V_1 A_1^2 V_1^T - V_2 A_2^2 V_2^T)A, \\ C &= -A(V_1 A_1^3 V_1^T - V_2 A_2^3 V_2^T)A + B \Gamma B, \end{aligned}$$

is hyperbolic and has eigenpairs  $(\lambda_k, v_k)$ ,  $k = 1: 2n$  [Al-Ammari and Tisseur 2011], [Guo et al. 2009a]. The quadratic  $Q(\lambda)$  has the property that  $A$  is positive definite and  $-Q(\mu)$  is positive definite for all  $\mu \in (\lambda_{\max}(A_2), \lambda_{\min}(A_1))$ . If  $\lambda_{\max}(A) < 0$  then  $B$  and  $C$  are positive definite and  $Q(\lambda)$  is overdamped.

`gen_tpal2` {pep,qep,real,T-palindromic,parameter-dependent,scalable,random}. This is a real T-palindromic quadratic matrix polynomial generated from a given set of eigenvalues

$$\lambda_{2j-1} = \cos t_j + i \sin t_j, \quad \lambda_{2j} = \bar{\lambda}_{2j-1}, \quad t_j \in (0, \pi), \quad j = 1: n, \quad (4)$$

lying on the unit circle. Let

$$J = \text{diag} \left( \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \beta_n \\ -\beta_n & 0 \end{bmatrix} \right), \quad i\beta_j = \frac{\lambda_{2j-1} - 1}{\lambda_{2j-1} + 1} \in \mathbb{R}, \quad (5)$$

$$S = \text{diag} \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \in \mathbb{R}^{2n \times 2n}, \quad X = [X_1 \quad X_1 H] \Pi \in \mathbb{R}^{n \times 2n},$$

where  $X_1 \in \mathbb{R}^{n \times n}$  is nonsingular,  $H \in \mathbb{R}^{n \times n}$  is a symmetric matrix, and  $\Pi$  is a permutation matrix such that  $\Pi^T S \Pi = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ . Then the  $n \times n$  real quadratic  $\tilde{Q}(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  with

$$A_2 = (X J S X^T)^{-1}, \quad A_1 = -A_2 X J^2 S X^T A_2, \quad A_0 = -A_2 (X J^2 S X^T A_1 + X J^3 S X^T A_2) \quad (6)$$

is real  $T$ -even with eigenvalues  $\pm i\beta_j$ ,  $j = 1:n$ . Finally,

$$Q(\lambda) = (\lambda + 1)^2 \tilde{Q}\left(\frac{\lambda - 1}{\lambda + 1}\right) = \lambda^2(A_2 + A_1 + A_0) + \lambda(-2A_2 + 2A_0) + (A_2 - A_1 + A_0) \quad (7)$$

is real  $T$ -palindromic with eigenvalues  $\lambda_j$ ,  $j = 1:2n$  (see Al-Ammari [2011]).

`gen_tantipal2` {pep,qep,real,T-anti-palindromic,parameter-dependent,scalable,random}. This is a real  $T$ -anti-palindromic quadratic matrix polynomial generated from a given set of eigenvalues lying on the unit circle as in (4). Let  $J$  be as in (5) and

$$S = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_1 U \end{bmatrix},$$

where  $X_1 \in \mathbb{R}^{n \times n}$  is nonsingular and  $U \in \mathbb{R}^{n \times n}$  is orthogonal. Then the  $n \times n$  real quadratic  $\tilde{Q}(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  with matrix coefficients as in (6) is real  $T$ -odd with eigenvalues  $\pm i\beta_j$ ,  $j = 1:n$ . Finally,  $Q(\lambda)$  in (7) is real  $T$ -anti-palindromic with eigenvalues  $\lambda_j$ ,  $j = 1:2n$  (see Al-Ammari [2011]).

`gun` {nep,sparse}. This nonlinear eigenvalue problem models a radio-frequency gun cavity. The eigenvalue problem is of the form

$$F(\lambda)x = [K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2]x = 0,$$

where  $M, K, W_1, W_2$  are real symmetric matrices of size  $9956 \times 9956$ .  $K$  is positive semidefinite and  $M$  is positive definite. In this example  $\sigma_1 = 0$  and  $\sigma_2 = 108.8774$ . The eigenvalues of interest are the  $\lambda$  for which  $\lambda^{1/2}$  is close to 146.71 [Liao 2007, p. 59].

`hadeler` {nep,real,symmetric,scalable}. This nonlinear eigenvalue problem has the form

$$F(\lambda)x = [(e^\lambda - 1)A_2 + \lambda^2 A_1 - \alpha A_0]x = 0,$$

where  $A_2, A_1, A_0 \in \mathbb{R}^{n \times n}$  are symmetric and  $\alpha$  is a scalar parameter [Haderler 1967]. This problem satisfies a generalized form of overdamping condition that ensures the existence of a complete set of eigenvectors [Ruhe 1973].

`hospital` {pep,qep,real}. This is a  $24 \times 24$  quadratic polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$ , with  $M = I_{24}$ , arising in the study of the Los Angeles University Hospital building [Chahlaoui and Van Dooren 2002], [Chahlaoui and Van Dooren 2005]. There are 8 floors, each with 3 degrees of freedom.

`intersection` {pep,qep,real}. This  $10 \times 10$  quadratic polynomial arises in the problem of finding the intersection between a cylinder, a sphere, and a plane described by the equations

$$\begin{aligned} f_1(x, y, z) &= 1.6\text{e-}3 x^2 + 1.6\text{e-}3 y^2 - 1 = 0, \\ f_2(x, y, z) &= 5.3\text{e-}4 x^2 + 5.3\text{e-}4 y^2 + 5.3\text{e-}4 z^2 + 2.7\text{e-}2 x - 1 = 0, \\ f_3(x, y, z) &= -1.4\text{e-}4 x + 1.0\text{e-}4 y + z - 3.4\text{e-}3 = 0. \end{aligned} \quad (8)$$

Use of the Macaulay resultant leads to the QEP  $Q(x)v = 0$ , where

$$\begin{aligned} Q(x)v &= [yf_1 \quad zf_1 \quad f_1 \quad yf_2 \quad zf_2 \quad f_2 \quad yzf_3 \quad yf_3 \quad zf_3 \quad f_3]^T = (x^2 A_2 + x A_1 + A_0)v, \\ v &= [y^3 \quad y^2 z \quad y^2 \quad yz^2 \quad z^3 \quad z^2 \quad yz \quad y \quad z \quad 1]^T. \end{aligned}$$

The matrix  $A_2$  is singular and the QEP has only four finite eigenvalues: two real and two complex. Let  $(\lambda_i, v_i)$ ,  $i = 1, 2$  be the two real eigenpairs. With the normalization  $v_i(10) = 1$ ,  $i = 1, 2$ ,  $(x_i, y_i, z_i) = (\lambda_i, v_i(8), v_i(9))$  are solutions of (8) [Manocha 1994].

`loaded_string` {rep,real,symmetric,parameter-dependent,sparse,scalable}. This rational eigenvalue problem arises in the finite element discretization of a boundary problem describing the eigenvibration of a string with a load of mass  $m$  attached by an elastic spring of stiffness  $k$ . It has the form

$$R(\lambda)x = \left( A - \lambda B + \frac{\lambda}{\lambda - \sigma} C \right) x = 0,$$

where the pole  $\sigma = k/m$ , and  $A > 0$  and  $B > 0$  are  $n \times n$  tridiagonal matrices defined by

$$A = \frac{1}{h} \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad B = \frac{h}{6} \begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},$$

and  $C = ke_n e_n^T$  with  $h = 1/n$  [Solov'ev 2006].

`metal_strip` {pep,qep,real}. Modelling the electronic behaviour of a metal strip using partial element equivalent circuits (PEEC's) results in the delay differential equation [Bellen et al. 1999]

$$\begin{cases} D_1 \dot{x}(t-h) + D_0 \dot{x}(t) = A_0 x(t) + A_1 x(t-h), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-h, 0), \end{cases}$$

where

$$A_0 = 100 \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_1 = 100 \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$D_1 = -\frac{1}{72} \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}, \quad D_0 = I, \quad \varphi(t) = [\sin(t), \sin(2t), \sin(3t)]^T.$$

Assessing the stability of this delay differential equation by the method in [Faßbender et al. 2008], [Jarlebring 2008] leads to the quadratic eigenproblem  $(\lambda^2 E + \lambda F + G)u = 0$  with

$$E = (D_0 \otimes A_1) + (A_0 \otimes D_1), \quad G = (D_1 \otimes A_0) + (A_1 \otimes D_0),$$

$$F = (D_0 \otimes A_0) + (A_0 \otimes D_0) + (D_1 \otimes A_1) + (A_1 \otimes D_1).$$

This problem is PCP-palindromic [Faßbender et al. 2008], i.e., there is an involutory matrix  $P$  such that  $E = PGP$  and  $F = PFP$ .

`mirror` {pep,real,random}. The  $9 \times 9$  quartic matrix polynomial  $\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$  is obtained from a homography-based method for calibrating a central catadioptric vision system, which can be built from a perspective camera with a hyperbolic mirror or an orthographic camera with a parabolic mirror [Zhang and Li 2008].  $A_0$  and  $A_4$  have only two nonzero columns, so there are at least 7 infinite eigenvalues and 7 zero eigenvalues.

`mobile_manipulator` {pep,qep,real}. This is a  $5 \times 5$  quadratic matrix polynomial arising from modelling a two-dimensional three-link mobile manipulator as a time-invariant descriptor control system [Byers et al. 1998, Ex. 14], [Bunse-Gerstner et al. 1999]. The system in its second-order form is

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t),$$

$$y(t) = Cx(t),$$

where the coefficient matrices are  $5 \times 5$  and of the form

$$M = \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix},$$

with

$$M_0 = \begin{bmatrix} 18.7532 & -7.94493 & 7.94494 \\ -7.94493 & 31.8182 & -26.8182 \\ 7.94494 & -26.8182 & 26.8182 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1.52143 & -1.55168 & 1.55168 \\ 3.22064 & 3.28467 & -3.28467 \\ -3.22064 & -3.28467 & 3.28467 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 67.4894 & 69.2393 & -69.2393 \\ 69.8124 & 1.68624 & -1.68617 \\ -69.8123 & -1.68617 & -68.2707 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadratic  $Q(\lambda) = \lambda^2 M + \lambda D + K$  is close to being nonregular [Byers et al. 1998], [Higham and Tisseur 2002].

`omnicam1` {pep,qep,real}. This is a  $9 \times 9$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  arising from a model of an omnidirectional camera (one with angle of view greater than 180 degrees) [Mičušík and Pajdla 2003]. The matrix  $A_0$  has one nonzero column,  $A_1$  has 5 nonzero columns and rank 5, while  $A_2$  has full rank. The eigenvalues of interest are the real eigenvalues of order 1.

`omnicam2` {pep,qep,real}. The description of `omnicam1` applies to this problem, too, except that the quadratic is  $15 \times 15$ .

`orr-sommerfeld` {pep,parameter-dependent,scalable}. This example is a quartic polynomial eigenvalue problem arising in the spatial stability analysis of the Orr-Sommerfeld equation [Tisseur and Higham 2001]. The Orr-Sommerfeld equation is a linearization of the incompressible Navier-Stokes equations in which the perturbations in velocity and pressure are assumed to take the form  $\Phi(x, y, t) = \phi(y)e^{i(\lambda x - \omega t)}$ , where  $\lambda$  is a wavenumber and  $\omega$  is a radian frequency. For a given Reynolds number  $R$ , the Orr-Sommerfeld equation may be written

$$\left[ \left( \frac{d^2}{dy^2} - \lambda^2 \right)^2 - iR \left\{ (\lambda U - \omega) \left( \frac{d^2}{dy^2} - \lambda^2 \right) - \lambda U'' \right\} \right] \phi = 0. \quad (9)$$

In spatial stability analysis the parameter is  $\lambda$ , which appears to the fourth power in (9), so we obtain a quartic polynomial eigenvalue problem. The quartic is constructed using a Chebyshev spectral discretization. The eigenvalues  $\lambda$  of interest are those closest to the real axis and  $\text{Im}(\lambda) > 0$  is needed for stability. The default values  $R = 5772$  and  $\omega = 0.26943$  correspond to the critical neutral point corresponding to  $\lambda$  and  $\omega$  both real for minimum  $R$  [Bridges and Morris 1984], [Orszag 1971].

`pdde_stability` {qep,pep,scalable,parameter-dependent,sparse,symmetric}. This problem arises from the stability analysis of a partial delay-differential equation (PDDE) [Faßbender et al. 2008], [Jarlebring 2008, Ex. 3.22]. Discretization gives rise to a time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2),$$

where  $A_0 \in \mathbb{R}^{n \times n}$  is tridiagonal and  $A_1, A_2 \in \mathbb{R}^{n \times n}$  are diagonal with

$$(A_0)_{kj} = \begin{cases} -2(n+1)^2/\pi^2 + a_0 + b_0 \sin(j\pi/(n+1)) & \text{if } k = j, \\ (n+1)^2/\pi^2 & \text{if } |k - j| = 1, \end{cases}$$

$$(A_1)_{jj} = a_1 + b_1 \frac{j\pi}{n+1} \left(1 - e^{-\pi(1-j/(n+1))}\right),$$

$$(A_2)_{jj} = a_2 + b_2 \frac{j\pi^2}{n+1} (1 - j/(n+1)).$$

Here, the  $a_k$  and  $b_k$  are real scalar parameters and  $n \in \mathbb{N}$  is the number of uniformly spaced interior grid points in the discretization of the PDDE. Asking for the delays  $h_1, h_2$  such that the delay system is stable leads to the quadratic eigenvalue problem  $(\lambda^2 E + \lambda F + G)v = 0$  of dimension  $n^2 \times n^2$  with

$$E = I \otimes A_2, \quad F = (I \otimes (A_0 + e^{-i\varphi_1} A_1)) + ((A_0 + e^{i\varphi_1} A_1) \otimes I), \quad G = A_2 \otimes I,$$

where  $i$  is the imaginary unit and  $\varphi_1 \in [-\pi, \pi]$  is a parameter. (To answer the stability question, the QEP has to be solved for many values of  $\varphi_1$ .)

Following [Jarlebring 2008], [Faßbender et al. 2008] the default values are

$$n = 20, \quad a_0 = 2, \quad b_0 = 0.3, \quad a_1 = -2, \quad b_1 = 0.2, \quad a_2 = -2, \quad b_2 = -0.3, \quad \varphi_1 = -\pi/2.$$

This problem is PCP-palindromic [Faßbender et al. 2008], i.e., there is an involutory matrix  $P$  such that  $E = P\bar{G}P$  and  $F = P\bar{F}P$ . Moreover, only the four eigenvalues on the unit circle are of interest in the application. The exact corresponding eigenvectors can be written as  $x_j = u_j \otimes v_j$  for  $j = 1:4$ .

`planar_waveguide` {pep, real, symmetric, scalable}. This  $129 \times 129$  quartic matrix polynomial  $P(\lambda) = \lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$  arises from a finite element solution of the equation for the modes of a planar waveguide using piecewise linear basis functions  $\phi_i, i = 0:128$ . The coefficient matrices are defined by

$$A_1 = \frac{\delta^2}{4} \text{diag}(-1, 0, 0, \dots, 0, 0, 1), \quad A_3 = \text{diag}(1, 0, 0, \dots, 0, 0, 1),$$

$$A_0(i, j) = \frac{\delta^4}{16} (\phi_i, \phi_j) \quad A_2(i, j) = (\phi'_i, \phi'_j) - (q\phi_i, \phi_j) \quad A_4(i, j) = (\phi_i, \phi_j).$$

Thus  $A_1$  and  $A_3$  are diagonal, while  $A_0, A_2$ , and  $A_4$  are tridiagonal. The parameter  $\delta$  describes the difference in refractive index between the cover and the substrate of the waveguide;  $q$  is a function used in the derivation of the variational formulation and is constant in each layer [Stowell 2010], [Stowell and Tausch 2010]. This particular waveguide has been studied in the literature in connection with other solution methods [Chilwell and Hodgkinson 1984], [Petráček and Singh 2002].

`plasma_drift` {pep}. This cubic matrix polynomial of dimension 128 or 512 results from the modeling of drift instabilities in the plasma edge inside a Tokamak reactor [Tokar et al. 2005]. It is of the form  $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ , where  $A_0$  and  $A_1$  are complex,  $A_2$  is complex symmetric, and  $A_3$  is real symmetric. The desired eigenpair is the one whose eigenvalue has the largest imaginary part.

`power_plant` {pep, qep, symmetric, parameter-dependent}. This is a QEP  $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$  describing the dynamic behaviour of a nuclear power plant simplified into an eight-degrees-of-freedom system [Itoh 1973], [Tisseur and Meerbergen 2001]. The mass matrix  $M$  and damping matrix  $D$  are real symmetric and the stiffness matrix has the form  $K = (1 + i\mu)K_0$ , where  $K_0$  is real symmetric (hence

$K = K^T$  is complex symmetric). The parameter  $\mu$  describes the hysteretic damping of the problem. The matrices are badly scaled.

**qep1** {pep,qep,real,solution}. This is a  $3 \times 3$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  from Tisseur and Meerbergen [2001, p. 250] with

$$A_2 = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0 = I.$$

The six eigenpairs  $(\lambda_k, x_k)$ ,  $k = 1: 6$ , are given by

$k$	1	2	3	4	5	6
$\lambda_k$	1/3	1/2	1	$i$	$-i$	$\infty$
$x_k$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Note that  $x_1$  is an eigenvector for both of the distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ .

**qep2** {pep,qep,real,solution}. This is the  $3 \times 3$  quadratic matrix polynomial [Tisseur and Meerbergen 2001, p. 256]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues are  $\lambda_1 = -1$ ,  $\lambda_2 = \lambda_3 = \lambda_4 = 1$ , and  $\lambda_5 = \lambda_6 = \infty$ . The Jordan structure is given by

$$X_F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_F = \text{diag} \left( -1, 1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

for the finite eigenvalues and and

$$X_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for the infinite eigenvalues (see Gohberg et al. [2009] or Tisseur and Meerbergen [2001, Sec. 3.6] for definitions of Jordan structure).

**qep3** {pep,qep,real,parameter-dependent,solution}. This is a  $3 \times 3$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  from Dedieu and Tisseur [2003, p. 89] with

$$A_2 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1 - \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 2 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

The eigenpairs  $(\lambda_k, x_k)$ ,  $k = 1: 6$ , are given by

$k$	1	2	3	4	5	6
$\lambda_k$	0	1	$1 + \epsilon$	2	3	$\infty$
$x_k$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \frac{\epsilon-1}{\epsilon+1} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For the default value of the parameter,  $\epsilon = -1 + 2^{-53/2}$ , the first and third eigenvalues are ill conditioned.

`qep4` {pep,qep,nonregular,nonsquare,real,solution}. This is the  $3 \times 4$  quadratic matrix polynomial [Byers et al. 2008, Ex. 2.5]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The eigensystem includes an eigenvalue  $\lambda_1 = 0$  with right eigenvectors  $[2 \ 1 \ 0 \ -1]^T$  and  $e_1$  and an eigenvalue  $\lambda = \infty$  with right eigenvector  $[0 \ 0 \ 1 \ 0]^T$ . The Jordan and Kronecker structure is fully described in Byers et al. [2008, Ex. 2.5].

`qep5` {pep,qep,nonregular,real}. This is the  $3 \times 3$  quadratic matrix polynomial [Van Dooren and Dewilde 1983, Ex. 1]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 4 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its Smith form [Gantmacher 1959] is given by

$$D(\lambda) = \begin{bmatrix} 1 & -1 & -1 \\ -\lambda & 1+\lambda & \lambda \\ 0 & -\lambda & 1 \end{bmatrix} Q(\lambda) \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and since  $\det(Q(\lambda)) = \det(D(\lambda)) \equiv 0$  this problem is nonregular.

`railtrack` {pep,qep,t-palindromic,sparse}. This is a T-palindromic quadratic matrix polynomial of size 1005:  $Q(\lambda) = \lambda^2 A^T + \lambda B + A$  with  $B = B^T$ . It stems from a model of the vibration of rail tracks under the excitation of high speed trains, discretized by classical mechanical finite elements [Hilliges 2004], [Hilliges et al. 2004], [Ipsen 2004], [Mackey et al. 2006]. This problem has the property that the matrix  $A$  is of the form

$$A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} \in \mathbb{C}^{1005 \times 1005},$$

where  $A_{21} \in \mathbb{C}^{201 \times 67}$ , that is,  $A$  has low rank ( $\text{rank}(A) = 67$ ). Hence this eigenvalue problem has many eigenvalues at zero and infinity.

`railtrack2` {pep,qep,t-palindromic,sparse,scalable,parameter-dependent}. This is a T-palindromic quadratic matrix polynomial of size  $705m \times 705m$ :  $Q(\lambda) = \lambda^2 A^T + \lambda B + A$  with

$$A = \begin{bmatrix} 0 & \dots & 0 & H_1 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} H_0 & H_1^T & & 0 \\ H_1 & H_0 & \ddots & \\ & \ddots & \ddots & H_1^T \\ 0 & & H_1 & H_0 \end{bmatrix} = B^T,$$

where  $H_0, H_1 \in \mathbb{C}^{705 \times 705}$  depend quadratically on a parameter  $\omega$ , whose default value is  $\omega = 1000$ . The default for the number of block rows and columns of  $A$  and  $B$  is  $m = 51$ . The structure of  $A$  implies that there are many eigenvalues at zero and infinity.

Like the problem `railtrack` this problem is from a model of the vibration of rail tracks, but here triangular finite elements are used for the discretization [Chu et al. 2008], [Guo and Lin 2010], [Huang et al. 2008]. The parameter  $\omega$  denotes the frequency of the external excitation force.

`relative_pose_5pt` {pep,real}. The cubic matrix polynomial  $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ ,  $A_i \in \mathbb{R}^{10 \times 10}$ , comes from the five point relative pose problem in computer



vision [Kukelova et al. 2008], [Kukelova et al. 2011]. In this problem the images of five unknown scene points taken with a camera with a known focal length from two distinct unknown viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The matrix  $A_3$  has one nonzero column,  $A_2$  has 3 nonzero columns and rank 3,  $A_1$  has 6 nonzero columns and rank 6, while  $A_0$  is of full rank. The solutions to the problem are obtained from the last three components of the finite eigenvectors of  $P$ .

`relative_pose_6pt` {pep,qep,real}. The quadratic matrix polynomial  $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ , where  $A_i \in \mathbb{R}^{10 \times 10}$ , comes from the six point relative pose problem in computer vision [Kukelova et al. 2008], [Kukelova et al. 2011]. In this problem the images of six unknown scene points taken with a camera of unknown focal length from two distinct unknown camera viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The solutions to the problem are obtained from the last three components of the finite eigenvectors of  $P$ .

`schrodinger` {pep,qep,real,symmetric,sparse}. The spectrum of this matrix polynomial is the second order spectrum, relative to a subspace  $\mathcal{L} \subset H^2(\mathbb{R})$ , of the Schrödinger operator  $Hf(x) = f''(x) + (\cos(x) - e^{-x^2})f(x)$  acting on  $L^2(\mathbb{R})$  [Boulton and Levitin 2007]. The subspace  $\mathcal{L}$  has been generated using fourth order Hermite elements on a uniform mesh on the interval  $[-49, 49]$ , subject to clamped boundary conditions. The corresponding quadratic matrix polynomial is given by  $K - 2\lambda C + \lambda^2 B$  where

$$K_{jk} = \langle Hb_j, Hb_k \rangle, \quad C_{jk} = \langle Hb_j, b_k \rangle \quad \text{and} \quad B_{jk} = \langle b_j, b_k \rangle.$$

Here  $\{b_k\}$  is a basis of  $\mathcal{L}$ . The matrices are of size 1998.

The essential spectrum of  $H$  consists of a set of bands separated by gaps. The end points of these bands are the Mathieu characteristic values. The presence of the short-range potential gives rise to isolated eigenvalues of finite multiplicity. The portion of the second order spectrum that lies in the box  $[-1/2, 2] \times [-10^{-1}, 10^{-1}]$  is very close to the spectrum of  $H$ .

`shaft` {pep,qep,real,symmetric,sparse}. The quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$ , with  $M, C, K \in \mathbb{R}^{400 \times 400}$ , comes from a finite element model of a shaft on bearing supports with a damper [Kowalski 2000, Ex. 5.6]. The matrix  $M$  has rank 199 and so contributes a large number of infinite eigenvalues.  $C$  has a single nonzero element, in the (20, 20) position. The coefficients  $M$ ,  $C$ , and  $K$  are very sparse.

`sign1` {pep,qep,hermitian,parameter-dependent,scalable}. The spectrum of this quadratic matrix polynomial is the second order spectrum of the linear operator  $Mf(x) = \text{sign}(x)f(x) + a\hat{f}(0)$  acting on  $L^2(-\pi, \pi)$  with respect to the Fourier basis  $\mathcal{B}_n = \{e^{-inx}, \dots, 1, \dots, e^{inx}\}$ , where  $\hat{f}(0) = (1/2\pi) \int_{-\pi}^{\pi} f(x) dx$  [Boulton 2007]. The corresponding QEP is given by  $K_n - 2\lambda C_n + \lambda^2 I_n$  where

$$K_n = \Pi_n M^2 \Pi_n, \quad C_n = \Pi_n M \Pi_n$$

and  $I_n$  is the identity matrix of size  $2n + 1$ . Here  $\Pi_n$  is the orthogonal projector onto  $\text{span}(\mathcal{B}_n)$ .

As  $n$  increases, the limit set of the second order spectrum is the unit circle, together with two real points:  $\lambda_{\pm}$ . The intersection of this limit set with the real line is the spectrum of  $M$ . The points  $\lambda_{\pm}$  comprise the discrete spectrum of  $M$ .

`sign2` {pep,qep,hermitian,parameter-dependent,scalable}. This problem is analogous to problem `sign1`, the only difference being that the operator is  $Mf(x) = (2\sin(x) + \text{sign}(x))f(x) + a\hat{f}(0)$ .

Near the real line, the second order spectrum accumulates at  $[-3, -1] \cup [1, 3] \cup \{\lambda_{\pm}\}$  as  $n$  increases. The two accumulation points  $\lambda_{\pm} \approx \{-0.7674, 3.5796\}$  are the discrete spectrum of  $M$ .

`sleeper` {pep,qep,real,symmetric,sparse,scalable,proportionally-damped,solution}. This QEP describes the oscillations of a rail track resting on sleepers [Lancaster and Rózsa 1996]. The QEP has the form

$$Q(\lambda) = \lambda^2 I + \lambda(I + A^2) + A^2 + A + I,$$

where  $A$  is the circulant matrix with first row  $[-2, 1, 0, \dots, 0, 1]$ . The eigenvalues of  $A$  and corresponding eigenvectors are explicitly given as

$$\mu_k = -4 \sin^2 \left( \frac{(k-1)\pi}{n} \right), \quad x_k(j) = \frac{1}{\sqrt{n}} \exp \left( \frac{-2i\pi(j-1)(k-1)}{n} \right), \quad k = 1:n.$$

The eigenvalues of  $Q$  can be determined from the scalar equations

$$\lambda^2 + \lambda(1 + \mu_k^2) + (1 + \mu_k + \mu_k^2) = 0.$$

Due to the symmetry, manifested in  $\sin(\pi - \theta) = \sin(\theta)$ , there are several multiple eigenvalues.

`speaker_box` {pep,qep,real,symmetric}. The quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$ , with  $M, C, K \in \mathbb{R}^{107 \times 107}$ , is from a finite element model of a speaker box that includes both structural finite elements, representing the box, and fluid elements, representing the air contained in the box [Kowalski 2000, Ex. 5.5]. The matrix coefficients are highly structured and sparse. There is a large variation in the norms:  $\|M\|_2 = 1$ ,  $\|C\|_2 = 5.7 \times 10^{-2}$ ,  $\|K\|_2 = 1.0 \times 10^7$ .

`spring` {pep,qep,real,symmetric,proportionally-damped,parameter-dependent,sparse,scalable}. This is a QEP  $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$  arising from a linearly damped mass-spring system [Tisseur 2000]. The damping constants for the dampers and springs connecting the masses to the ground, and those for the dampers and springs connecting adjacent masses, are parameters. For the default choice of the parameters, the  $n \times n$  matrices  $K$ ,  $C$ , and  $M$  are

$$M = I, \quad C = 10T, \quad K = 5T, \quad T = \begin{bmatrix} 3 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 3 \end{bmatrix}.$$

`spring_dashpot` {pep,qep,real,parameter-dependent,sparse,scalable,random}. Gotts [Gotts 2005] describes a QEP arising from a finite element model of a linear spring in parallel with Maxwell elements (a Maxwell element is a spring in series with a dashpot). The quadratic matrix polynomial is  $Q(\lambda) = \lambda^2 M + \lambda D + K$ , where the mass matrix  $M$  is rank deficient and symmetric, the damping matrix  $D$  is rank deficient and block diagonal, and the stiffness matrix  $K$  is symmetric and has arrowhead structure. This example reflects the structure only, since the matrices themselves are not from a finite element model but randomly generated to have the desired properties

of symmetry etc. The matrices have the form

$$M = \text{diag}(\rho \tilde{M}_{11}, 0), \quad D = \text{diag}(0, \eta_1 \tilde{K}_{11}, \dots, \eta_m \tilde{K}_{m+1, m+1}),$$

$$K = \begin{bmatrix} \alpha_\rho \tilde{K}_{11} & -\xi_1 \tilde{K}_{12} & \dots & -\xi_m \tilde{K}_{1, m+1} \\ -\xi_1 \tilde{K}_{12} & e_1 \tilde{K}_{22} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\xi_m \tilde{K}_{1, m+1} & 0 & 0 & e_m \tilde{K}_{m+1, m+1} \end{bmatrix},$$

where  $\tilde{M}_{ij}$  and  $\tilde{K}_{ij}$  are element mass and stiffness matrices,  $\xi_i$  and  $e_i$  measure the spring stiffnesses, and  $\rho$  is the material density.

**surveillance** {pep, qep, real, nonsquare, nonregular}. This is a  $21 \times 16$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  arising from calibration of a surveillance camera using a human body as a calibration target [Mičušík and Pajdla 2010]. The eigenvalue represents the focal length of the camera. This particular data set is synthetic and corresponds to a  $600 \times 400$  pixel camera.

**time\_delay** {nep, real}. This  $3 \times 3$  nonlinear matrix function has the form  $R(\lambda) = -\lambda I_3 + A_0 + A_1 \exp(-\lambda)$  with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b_3 & -b_2 & -b_1 \end{bmatrix},$$

and is the characteristic equation of a time-delay system with a single delay and constant coefficients [Jarlebring 2011], [Jarlebring and Michiels 2010], [Jarlebring and Michiels 2011]. The problem has a double non-semisimple eigenvalue  $\lambda = 3\pi i$ .

**wing** {pep, qep, real}. This example is a  $3 \times 3$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$  from Frazer et al. [1938, Sec. 10.11], with numerical values modified as in Lancaster [1966, Sec. 5.3]. The eigenproblem for  $Q(\lambda)$  arose from the analysis of the oscillations of a wing in an airstream. The matrices are

$$A_2 = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

**wiresaw1** {pep, qep, real, t-even, gyroscopic, parameter-dependent, scalable}. This gyroscopic QEP arises in the vibration analysis of a wiresaw [Wei and Kao 2000]. It takes the form  $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$ , where the  $n \times n$  coefficient matrices are defined by

$$M = I_n/2, \quad K = \text{diag}_{1 \leq j \leq n} (j^2 \pi^2 (1 - v^2)/2),$$

and

$$C = -C^T = (c_{jk}), \quad \text{with} \quad c_{jk} = \begin{cases} \frac{4jk}{j^2 - k^2} v, & \text{if } j + k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here,  $v$  is a real nonnegative parameter corresponding to the speed of the wire. Note that for  $0 < v < 1$ ,  $K$  is positive definite and the quadratic

$$G(\lambda) := -Q(-i\lambda) = \lambda^2 M + \lambda(iC) - K$$

is hyperbolic (but not overdamped).

`wiresaw2` {pep,qep,real,parameter-dependent,scalable}. When the effect of viscous damping is added to the problem in `wiresaw1`, the corresponding quadratic has the form [Wei and Kao 2000]

$$\tilde{Q}(\lambda) = \lambda^2 M + \lambda(C + \eta I) + K + \eta C,$$

where  $M$ ,  $C$ , and  $K$  are the same as in `wiresaw1` and the damping parameter  $\eta$  is real and nonnegative.

#### 4. DESIGN OF THE TOOLBOX

The problems in the NLEVP collection are accessed via a single MATLAB function `nlevp`, which is modelled on the MATLAB gallery function. This function calls those that actually generate the problems, which reside in a private directory located within the `nlevp` directory. This approach avoids the problem of name clashes with existing MATLAB functions and also provides an elegant interface to the collection.

All problems have input parameters comprising the problem name followed by (where applicable) the dimension and other parameters, and the coefficient matrices defining the problem (as specified in Section 2.1) are returned in a cell array. To illustrate, the following example sets up the `omnicam2` problem, finds its eigenvalues and eigenvectors with `polyeig`, and then prints the largest modulus of the eigenvalues:

```
>> coeffs = nlevp('omnicam2')
coeffs =
    [15x15 double]    [15x15 double]    [15x15 double]
>> [X,e] = polyeig(coeffs{:}); max(abs(e))
ans =
    3.6351e-001
```

The nonlinear function  $F(\lambda)$  in (3) can be evaluated by calling `nlevp` with `eval` as its first argument. This is useful for evaluating the residual of an approximate eigenpair, for example:

```
>> lam = e(end); x = X(:,end); Fx = nlevp('eval','omnicam2',lam)*x; norm(Fx)
ans =
    5.8137e-032
```

The second output argument from `nlevp` is a function handle that enables the nonlinear scalar functions  $f_i(\lambda)$  in (3) and their derivatives to be evaluated. This facilitates the use of numerical methods that require derivatives, especially for the non-polynomial problems, for which obtaining the derivatives can be nontrivial. For example, the following code evaluates  $f_i(0.5)$ ,  $i = 1:3$ , and the first two derivatives (denoted `fp`, `fpp`), for the fiber problem:

```
>> [coeffs,fun] = nlevp('fiber');
>> [f,fp,fpp] = fun(0.5)
f =
    1.0000e+000   -5.0000e-001   -7.0746e-001
fp =
           0   -1.0000e+000   -7.0725e-001
fpp =
           0           0    7.0696e-001
```

Problems and their properties are stored in a simple database made from cell arrays. The database is accessed with the `nlevp_query` function in the private directory,

which is invoked using the query argument to `nlevp`. For example, the properties for the butterfly problem are returned in a cell array by the following call (whose syntax illustrates the command/function duality of MATLAB [Higham and Higham 2005, Sec. 7.5]):

```
>> nlevp query butterfly
ans =
    'pep'
    'real'
    'parameter-dependent'
    'T-even'
    'scalable'
```

A more sophisticated example finds the names of all PEPs of degree 3 or higher:

```
>> pep = nlevp('query','pep'); qep = nlevp('query','qep');
>> pep_cubic_plus = setdiff(pep,qep)
pep_cubic_plus =
    'butterfly'
    'mirror'
    'orr_sommerfeld'
    'planar_waveguide'
    'plasma_drift'
    'relative_pose_5pt'
```

The cell array `pep_cubic_plus` can then easily be used to extract these problems. For example, the first problem in `pep_cubic_plus` can be solved using

```
coeffs = nlevp(pep_cubic_plus{1}); [X,e] = polyeig(coeffs{:});
```

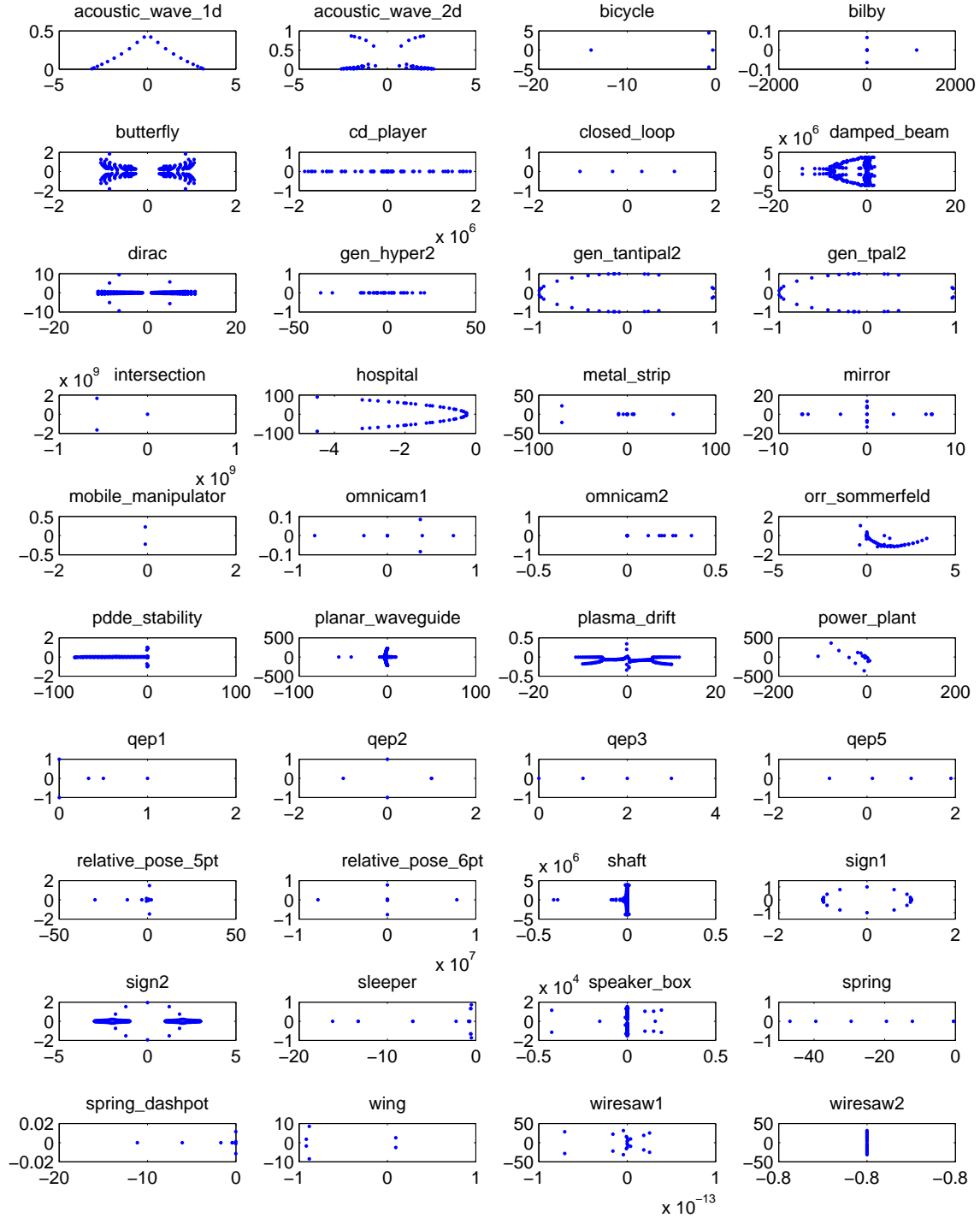
Table V and VI were generated automatically in MATLAB using appropriate `nlevp('query',...)` calls.

The toolbox function `nlevp_example` provides a test that the toolbox is correctly installed. It solves all the PEPs in the collection of dimension less than 500 using MATLAB's `polyeig` and then plots the eigenvalues. It produces Figure 1 and output to the command window that begins as follows:

```
NLEVP contains 52 problems in total,
of which 47 are polynomial eigenvalue problems (PEPs).
Run POLYEIG on the PEP problems of dimension at most 500:
```

Problem	Dim	Max and min magnitude of eigenvalues
-----	---	-----
acoustic_wave_1d	10	3.14e+000, 4.59e-001
acoustic_wave_2d	30	2.61e+000, 6.83e-001
bicycle	2	1.41e+001, 3.23e-001
bilby	5	Inf, 4.84e-016
butterfly	64	2.01e+000, 3.59e-001
cd_player	60	1.87e+006, 2.23e-004
closed_loop	2	1.07e+000, 3.31e-001
concrete	2472	is a PEP but is too large for this test.
...		

The `nlevp_example.m` function can be used as a template by the user wishing to test a given solver on subsets of the NLEVP problems.

Fig. 1. Eigenvalue plots for PEP problems produced by `nlevp_example.m`.

The toolbox function `nlevp_test.m` automatically tests that the problems in the collection have the claimed properties. It is primarily intended for use by the developers as new problems are added, but it can also be used as a test for correctness of the installation. While many of the tests are straightforward, some are less so. For example, we test for hyperbolicity of a Hermitian matrix polynomial by computing the eigensystem and checking the types of the eigenvalues, using a characterization in [Al-Ammari and Tisseur 2011, Thm. 3.4, P1]. To test for proportional damping we use necessary and sufficient conditions from Lancaster and Zaballa [2009, Thms. 2, 4]. We reproduce part of the output:

```
>> nlevp_test
Testing the NLEVP collection
Testing generation of all problems
Testing T-palindromicity
Testing *-palindromicity
...
Testing proportionally damping
Testing given solutions
NLEVP collection tests completed.
*** Errors: 0
```

## 5. CONCLUSIONS

The NLEVP collection demonstrates the tremendous variety of applications of nonlinear eigenvalue problems and provides representative problems for testing in the form of a MATLAB toolbox. Version 1.0 of the toolbox was released in 2008 and the current version is 3.0. The toolbox has already proved useful in our own work and that of others [Asakura et al. 2010], [Betcke 2008], [Betcke and Kressner 2011], [Grammont et al. 2011], [Guo et al. 2009b], [Hammarling et al. 2011], [Jarlebring et al. 2010], [Su and Bai 2011], [Tisseur et al. 2011] and we hope it will find broad use in developing, testing, and comparing new algorithms. By classifying important structural properties of nonlinear eigenvalue problems, and providing examples of these structures, this work should also be useful in guiding theoretical developments.

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