MATH41122 - 2008/2009

General Information

- Title: Differential Geometry
- Unit code: MATH41122
- Credits: 15
- Prerequisites: MATH20222 *Introduction to Geometry;* MATH31061/41061 *Differentiable Manifolds;* MATH31051 Introduction to Topology (optional): may be beneficial, but not required
- Co-requisite units:
- School responsible: Mathematics
- Member of staff responsible: Dr. Ted Voronov

Specification

Aims

The course aims to introduce the basic ideas of differential geometry.

Brief Description of the unit

This course unit introduces the main notions of modern differential geometry such as **connection** and **curvature** . It builds on the course unit MATH31061/MATH41061 Differentiable Manifolds.

A central role in differential geometry is played by **fiber bundles** . A fiber bundle is a manifold (or a topological space) that locally looks like the product of one space called the *base* with another space called the *fiber*. The whole space is the union of copies of the fiber parametrized by the points of the base. A good example is the Möbius band which locally looks like the product of a piece of a circle S^1 with an interval I , but globally involves a "twist" making it different from the cylinder $S^1 \times I$. Particularly important are vector bundles, for which fibers are vector spaces. They make a natural language for describing fields appearing in physics. An example of a vector bundle is the tangent bundle of a manifold.

In order to differentiate vector fields on a manifold (and more generally, *sections* of vector bundles) one needs an extra structure known as a *connection* or *covariant derivative* . It often comes naturally in examples such as tangent vectors to surfaces in Euclidean space. In this case a covariant derivative can be defined as the usual derivative in the ambient Euclidean space followed by the orthogonal projection onto the tangent plane. The curvature of a connection is a new phenomenon which does not occur for ordinary derivatives. Its simplest example is the 'internal curvature' of a surface, which is responsible for the fact that it is impossible to map a region of a sphere onto flat surface preserving distances.

The course revises classical differential geometry of curves and surfaces, considers applications, and touches on topology of fiber bundles.

Learning Outcomes

On completion of this unit successful students will be able to:

- deal with various examples of vector bundles;
- have familiarity with local frames, sections and transition functions;
- work practically with covariant derivative, local connection and curvature forms;
- appreciate the basic idea of differential-geometric characteristic classes;
- apply the methods of differential geometry to other areas.

Future topics requiring this course unit

Differential geometry is one of the pillars of modern mathematics. It is widely used in many applications outside mathematics, including physics and engineering. The following course unit may be considered as related:

MATH41101 Towards toric Topoogy

Syllabus

1. **Introduction.** The problem of a 'covariant' differentiation of vector fields. Examples. Idea of a fiber bundle. Non-trivial bundles.
2. **Vector bundles** Definition of a fiber bundle. Transition functions and the cocycle property. Vector bundles. Sections. Local frames.
3. **Connection on a vector bundle.** Axioms of a covariant derivative. Local connection 1-form. Existence of connection. Connection defined by a projector.
4. **Metric and connection.** Metric (scalar or inner product) on a vector bundle. Connection compatible with metric. Property of local connection 1-forms. Existence. Examples.
5. **Connections on manifolds.** Christoffel symbols in coordinate and non-coordinate bases. Torsion tensor. Symmetric connections. Levi-Civita theorem. Christoffel formulas.
6. **Curves and surfaces.** Tangent and normal bundles. Osculating planes, Frenet basis and Frenet equations for curves in $\boldsymbol{R}^n$. Derivation formulas for a hypersurface. "Normal" and "geodesic" curvature of a curve on a surface.
7. **Curvature of a connection.** *Theorema Egregium* and its meaning. Definition of the curvature 2-form for a connection on a vector bundle. Properties and interpretations.
8. **Parallel transport.** Examples and the main idea. Equation of parallel transport. Multiplicative integral. Geodesic lines.
9. **Curvature and parallel transport.** Parallel transport over a closed contour. Rotation of a vector under the parallel transport on a surface. Excess of a geodesic triangle. Gauß-Bonnet theorem for triangulated surfaces.
10. **Characteristic classes.** Example: the first Chern class. Construction of characteristic classes from invariant polynomials.
11. **Classification of vector bundles.** Pull-back of vector bundles. Homotopy property. Embedding into a trivial bundle. Classifying spaces.

Textbooks

No particular textbook is followed. Students are advised to keep their own lecture notes. There are many good sources available treating various aspects of differential geometry on various levels and from different viewpoints. Below is a list of texts that may be useful. More can be found by searching library shelves.

- R. Abraham, J. E. Marsden, T. Ratiu. Manifolds, tensor analysis, and applications.
- B.A. Dubrovin, A.T. Fomenko, S.P. Novikov. Modern geometry, methods and applications.
- S. Kobayashi, K. Nomizu. Foundations of differential geometry.
- A. S. Mishchenko, A. T. Fomenko. A course of differential geometry and topology,
- S. Morita. Geometry of differential forms,
- R. Wells. Differential analysis on complex manifolds.

Teaching and learning methods

Two lectures per week with additional reading plus one weekly examples class. In addition students should expect to spend at least seven hours each week on private study for this course unit.

Assessment

Coursework (take home work, details to be confirmed); weighting 20%.
End of semester examination (3 hours); weighting 80%.

Arrangements