MATH31011/41011 - 2008/2009

General Information

- Title: Measure and Fractals
- Unit code: MATH31011/41011
- Credits: 10 (MATH31011), 15 (MATH41011)
- Prerequisites: MATH20101 or MATH20111; MATH20122 *Metric Spaces*
- Co-requisite units: None
- School responsible: Mathematics
- Members of staff responsible: Prof. Richard Sharp

Specification

Aims

To give an introduction to Lebesgue measure on the set of real numbers $\boldsymbol{R}$ and the concept of measure in general, indicating its role in the theory of integration. To introduce fractal sets in $\boldsymbol{R}$ and $\boldsymbol{R}^n$ and the concept of Hausdorff dimension, concentrating on simple examples.

Brief Description of the unit

A standard approach to integration on the real line, formalised by Riemann, is based on partitioning the domain into smaller intervals. (This theory was described in MATH20101 but is not a prerequisite for the course.) This approach works in many situations but there are simple examples for which it fails. In 1902, H. Lebesgue produced a better theory in which the key idea is to extend the notion of length from intervals to more complicated subsets of $\boldsymbol{R}$. This started an area of mathematics it its own right, called Measure Theory. Most generally, this is about how one may sensibly assign a size to members of a collection of sets.

An interesting class of sets in $\boldsymbol{R}$ and higher dimensions are called fractals. These often exhibit self-similarity and are complicated by not too complicated to study. In fact, they give examples of complicated sets to which the Lebesgue theory assigns zero measure and the key idea of Hausdorff dimension gives a more delicate way of quantifying their size.

This course will appeal to students who have enjoyed MATH20101 or MATH20111 and MATH20122. It will be useful to student taking probability course courses in years three and four since the ideas of measure theory have a central role in probability theory.

Learning Outcomes

On successful completion of this course unit students will

- understand how Lebesgue measure on $\boldsymbol{R}$ is constructed,
- understand the general concept of measure,
- understand how measures may be used to construct integrals,
- understand the notion of Hausdorff dimension of sets in $\boldsymbol{R}^n$,
- calculate Hausdorff dimension for Cantor sets and other self-similar examples.

Future topics requiring this course unit

It would be helpful for level 3 and 4 courses in probability.

Syllabus

1. Riemann's approach to integration. Countable and uncountable sets. The Middle Third Cantor set. [3 lectures]
2. Lesbesgue measure on $\boldsymbol{R}$, measurable sets, Borel sets, non-measurable sets. Informal discussion of extension to $\boldsymbol{R}^n$, concept of a measure space. [9]

3. Overview of integration theory. [2]
4. Hausdorff measure and dimension, properties of Hausdorff dimension. Examples of fractal sets (Cantor sets, von Koch curve, Sierpinski gasket) and self-similarity. Calculation of Hausdorff dimension. [8]

For MATH41011 the lectures will be enhanced by additional reading on integration including details of the construction of the integral and proof of its properties. Reading material will be provided.

Textbooks

- H. S. Bear, *A Primer of Lebesgue Integration*, Academic Press, 1995.
- M. Capinski and E. Kopp, *Measure, Integral and Probability*, Springer (SUMS Series), 1999.
- K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed, Wiley, 2003.

Teaching and learning methods

Two lectures and one examples class each week. In addition students should expect to spend at least four hours each week on private study for this course unit (seven hours for MATH41011).

Assessment

End of semester examination: two hours weighting 100% (MATH31011), three hours weighting 100% (MATH41011)

Arrangements