On-line course materials

MATH20401 - Partial Differential Equations and Vector Calculus A

Year: 2 - **Semester:** 1 - **Credit Rating:** 20

Requisites

Prerequisites
MATH10121 Calculus and Vectors A
MATH10131 Calculus and Vectors B

Aims

On completion of the course students should

- understand more about applied mathematics and see the importance of mathematical modelling,
- see that PDEs describe a number of important phenomena,
- understand that vector calculus has a fundamental importance in applied mathematics,
- understand that one can use analytical and/or numerical methods (usually a combination of both) in order to solve PDEs that arise in real world applications,
- be well prepared for future courses in applied mathematics and numerical analysis.

Brief Description

The first half of this course equips students with the fundamental tools required in order to solve simple partial differential equations (PDEs). This includes important aspects of vector calculus (curvilinear coordinates and integral theorems) as well as Fourier series and an understanding of how to classify PDEs and what this classification means physically. The method of characteristics is then introduced in order to solve First order quasi-linear PDEs. The second half of the course focuses on solving second order PDEs (mainly Laplaces equation, the heat equation and the wave equation), first analytically by employing separation of variables and then numerically by introducing the topic of finite difference methods.

Learning Outcomes

On completion of the course students should be able to

- partially differentiate functions of several variables;

- convert Cartesian coordinates into general curvilinear coordinates including circular cylindrical polar and spherical polar coordinates;
- compute line, surface and volume elements in curvilinear coordinates;
- evaluate line, surface and volume integrals in three dimensions;
- use grad, div and curl, understand their definitions and be able to convert them into curvilinear coordinate systems;
- understand and employ Gauss (Divergence) and Stokes theorems;
- understand suffix (index) notation and be able to write down a vector expression in suffix notation;
- classify partial differential equations (PDEs) and identify their type (elliptic, parabolic, hyperbolic, mixed);
- choose appropriate boundary and initial conditions for PDEs;
- recognize that the three fundamental second order PDEs model physical phenomena;
- solve first order quasi-linear PDEs using the method of characteristics;
- compute Fourier Series, Fourier sine and cosine series of piecewise continuous functions;
- understand and be able to apply Dirichlets theorem for convergence of Fourier series;
- use separation of variables to solve the heat, wave and Laplaces equation for problems in one and two spatial dimensions on various domains;
- understand how separation of variables leads to the study of "Special functions" including Bessel functions, Legendre polynomials and Chebyshev polynomials;
- use Frobenius method to solve second order PDEs;
- be able to identify a Sturm-Liouville eigenvalue problem and know its properties;
- understand how to develop a Finite difference method to solve a PDE and how to implement these for the heat equation;
- understand the importance of, and difference between, error, stability and convergence of numerical methods;
- use finite difference methods to solve convection-diffusion problems and improve them by using "upwind" finite differences;

Syllabus

- **Section 1: Introduction and motivation**. What are PDEs? Why study them? Some examples and applications. [2 lectures]
- **Section 2: Vector calculus in curvilinear coordinates**. Introduction to general formalism of switching from Cartesian to curvilinear coordinate systems. Basis vectors, line, surface and volume elements. Grad, div, curl and transforming to curvilinear coordinates. Surface and volume integrals in three dimensions. Suffix notation. Gauss (divergence) and Stokes theorems in three dimensions. [6 lectures]
- **Section 3: Classification of PDEs**. Classification as order, scalar/vector, homogeneous/ inhomogeneous, linear/semi-linear/quasi-linear/nonlinear. PDE type: 2nd order in two independent variables (elliptic, hyperbolic, parabolic, mixed), canonical forms, extension to n variables, 1st order system of PDEs. Characteristics. Cauchy problem, well-posedness, choice of boundary and initial conditions [3 lectures]
- **Section 4: First order PDEs**. Scalar first order pdes in two variables. Linear constant coefficient, DAlembert. Method of characteristics for semi-linear and quasi-linear equations. [5 lectures]
- **Section 5: Fourier series**. Motivation via trial separation of variables solution for homogeneous heat equation in 1D with homogeneous boundary conditions, general initial profile. General concepts of eigenvalues/eigenfunctions, orthogonality. Fourier series, sine and cosine series and associated Fourier (Dirichlet) theorem regarding piecewise-smooth functions, orthogonality. Differentiating and integrating. [4 lectures]
- **Section 6: Separation of variables for second order PDEs**. Separation of variables for homogeneous heat and wave equation in curvilinear coordinates with homogeneous BCs and inhomogeneous initial conditions. Separation of variables for Laplaces equation with inhomogeneous

BCs. Link with Sturm Liouville eigenvalue problems. Special functions: circular functions, Bessel functions, Legendre polynomials, Chebyshev polynomials, Frobenius method. [12 lectures]

- **Section 7: Numerical solution of PDEs**. Finite difference methods. Link with solutions obtained in Section 4, 6 and 7. Explicit and implicit schemes and the theta method, truncation error, stability and convergence, Crank Nicholson, convection-diffusion problems, upward differencing. [12 lectures]

Teaching & Learning Process (Hours Allocated To)

Lectures	Tutorials/ Example Classes	Practical Work/ Laboratory	Private Study	Total
44	22	0	134	200

Assessment and Feedback

- One test worth 20% in week 6 (on material covered in sections 2-4 of the syllabus).
- 3 hours end of semester examination on ALL material; worth 80%

Further Reading

- Vector analysis. Schaums outlines. Editors: M.R. Spiegel, S. Lipschutz and D. Spellman. 2nd edition. 2009
- Div, Grad, Curl and all that: an informal text on vector calculus. H.M.Schey. W.W. Norton and Co. 4th Edition. 2005.
- An introduction to partial differential equations. Y. Pinchover and J. Rubinstein. Cambridge University Press. 2005
- Applied Partial Differential Equations with Fourier Series and Boundary Value Problems. R. Haberman, Pearson, 5th edition, 2012
- Numerical Solution of Partial Differential Equations. K.W. Morton and D.F. Mayers, Cambridge University Press. 2nd Edition. 2005

Staff Involved

Dr William Parnell - Lecturer

Dr Simon Cotter - Lecturer

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